Harnack Inequalities on Graphs

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Quick sketch of the talk

1. Introduce notations and structures from graph theory
2. Look at the Harnack inequality for abelian homogeneous graphs
3. Introduce graph Ricci curvature
4. Look at the advances in Harnack inequalities after the advent of Ricci curvature as well as eigenvalue estimates
5. Finally look at the “Strong” Harnack inequality developed in 2017

Introduction: Harnack's Inequality for the Dirichlet Problem

Theorem (Harnack's inequality)

Let $\Omega \subset \mathbb{R}^n$ be a domain, and $\omega \subset \Omega$ a compact, connected subset of Ω. Let $u \geq 0$ be a nonnegative C^2 solution to the problem

$$-\Delta u = 0 \text{ in } \Omega,$$

i.e. harmonic. Then there exists a constant C, depending only on Ω, so that

$$\sup_{x\in\omega} u(x) \leq C \inf_{x\in\omega} u(x).$$

- More can be found on the importance of this inequality, and its generalizations, in any standard reference, e.g. Evans[5].
- The main idea is that harmonic functions have values that can be controlled by universal quantities depending on the domain. The goal of this talk is to review the work of researchers in spectral graph theory which focus on reworking Harnack's inequality to the setting of graph Laplacians.
- Harnack inequalities for graphs can be used for eigenvalue estimates. These can in turn help estimate converges rates of random walks on large networks.

Background: Graph notations

- In this talk we will be working with a graph of the form $G = (V, E)$ where V is a finite list of $n \geq 1$ **vertices**, and $E \subset \binom{V}{2}$ is a collection of undirected pairs of vertices called **edges**. We assume that there are no "loops," i.e., edges of the form $\{x, x\}$, nor are there multiple edges.
- G is called **connected** if you can connect each pair of vertices with a sequence of edges forming a path.
- The **degree** of a vertex $x \in V$, denoted d_x, is the number of vertices adjacent to x; we denote adjacency of vertices by $x \sim y$. The **maximal degree** in the graph is $d = \max_x d_x$ and the **diameter**, or maximal shortest-path distance between any two vertices, is D.
- We identify subsets $H \subset V$ with their induced subgraphs; an induced subgraph $H \subset G$ is called **strongly convex** if, given any two $x, y \in H$ the shortest path connecting x and y also lies in H.
- The function space of interest, in the conventional notation, is

$$V^{\mathbb{R}} := \{f : V \to \mathbb{R}\}$$

as a linear space isomorphic to $\mathbb{R}^n$. We can identify functions with column vectors in $\mathbb{R}^n$ having fixed an enumeration of the vertices. Some authors adopt the notation $\ell_2(V)$ with the inherited Hilbert structures, but we won't really need that here.

Background: Graph Laplacians

- There are many ways to "discretize" the Laplacian to the space of functions on graphs. The overarching theme is that a graph Laplacian operator should satisfy various properties enjoyed by the classical Laplacian, e.g., linearity, positivity, mean value property, maximum principle, symmetricity. But there are many ways to define it[1].
- The **graph Laplacian operator** of interest to us, denoted $\Delta : V^{\mathbb{R}} \to V^{\mathbb{R}}$, is defined by

$$-(\Delta f)(x) = \frac{1}{d_x} \sum_{\substack{y \\ y \sim x}} (f(x) - f(y))$$

This operator (or rather $-\Delta$) is linear, positive, symmetric.
- This version is normalized locally by the degree of each vertex; perhaps more common in the literature is the combinatorial graph Laplacian which omits this leading factor.
- A function $f \in V^{\mathbb{R}}$ is called a **harmonic eigenfunction** for $-\Delta$ with eigenvalue $\lambda \geq 0$ provided $-\Delta f = \lambda f$.

[1] See Wardetsky et al.[7] for a detailed treatment on this interesting problem

Homogeneous Graphs

- G is called a **homogeneous graph** with associated group Γ provided
  1. Γ acts by left action on V,
  2. For all $g \in \Gamma$, $\{gx, gy\} \in E$ if and only if $\{x, y\} \in E$
  3. For any two vertices $x, y \in V$ there is a $g = g(x, y) \in \Gamma$ so that $x = gy$.

  If Γ is abelian, G is called an **abelian homogeneous graph** with respect to Γ.
- We can identify V as the coset space $\Gamma/\mathcal{I}$ where $\mathcal{I} = \{g \in \Gamma : gv = v\}$ for a fixed vertex v, is the **isotropy group**. The Cayley graph of Γ is the special case where $\mathcal{I}$ is trivial[3].
- We require that there is a subset $K \subset \Gamma$, closed under inverses, such that for each vertex $x \in V$, K generates the neighborhood of x, i.e.

$$\{y \in V : y \sim x\} = \{kx : k \in K\}.$$

  We say K is an **invariant generating set** if it is invariant as a set in Γ under conjugation by elements of K.

Examples - homogeneous graphs

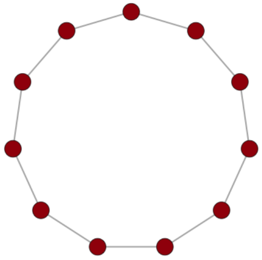

Figure: A cycle graph on 11 vertices

- Some examples of abelian homogeneous graphs include cycle graphs on n vertices, where the associated group is $\mathbb{Z}/n\mathbb{Z}$ and the generating set is $\{1, -1\} = \{1, n-1\} \subset \mathbb{Z}/n\mathbb{Z}$
- You can think of ± 1 as the "left" and "right" steps needed to generate local neighborhoods from a fixed point, and $\mathbb{Z}/n\mathbb{Z}$ as the steps needed to generate the graph globally from a fixed point.

Neumann eigenvalues

- For a strongly convex induced subgraph $S \subset G$, the Nuemann eigenvalue λ_S is defined via the Rayleigh quotient

$$\lambda_S = \inf_{f \in W} \frac{\sum_{\{x,y\} \in S \cup \partial S} (f(x) - f(y))^2}{k \sum_{x \in S} f(x)^2}$$

where $W \subset V^{\mathbb{R}}$ is the space of all functions for which $\sum_{x \in S} f(x) = 0$.

Lemma (Chung, Yau[3])

Let G be a abelian homogeneous invariant graph with associated group Γ and generating set K, with k elements. Let S be a strongly convex induced subgraph with Neumann eigenvalue λ_S and minimizing function $f \in V^{\mathbb{R}}$. Then the following two equations hold:

$$\begin{cases} (\Delta f)(x) = \lambda_S f(x) & x \in S \\ \sum_{\substack{a \in K \\ ax \in S}} (f(x) - f(ax)) = 0 & x \in \partial S \end{cases}$$

where ∂S is the vertex boundary of S.

- Note - the second condition is often referred to as the “vanishing normal derivative” condition; hence the name Neumann eigenvalue.

Theorem (Chung, Yau [3])

Let S be a strongly convex subgraph in an invariant homogeneous abelian graph G with generating set K of size k. Suppose $f : S \to \mathbb{R}$ is a harmonic eigenfunction of $-\Delta$ with eigenvalue $\lambda \geq 0$. Then for all $x \in S$ and $a \in K$, it holds

$$|f(x) - f(ax)|^2 \leq 8k\lambda \sup_{y \in S} |f|^2(y)$$

Theorem (Chung, Yau[3])

Let G be an abelian homogeneous invariant graph with generating set K on k elements, and S a strongly convex induced subgraph with Neumann eigenvalue λ_S. Then the following holds:

$$\lambda_S \geq \frac{1}{8kD^2}$$

where D is the diameter of S, i.e., the maximal shortest-path distance between two vertices in S.

Graph Ricci Curvature

- Discretizing Ricci curvature follows a similar route as graph Laplacian operators. The seminal resource is work by Bakry-Émery[1].
- This is done by defining a **first curvature operator**, which is the billinear operator defined $\Gamma : V^{\mathbb{R}} \times V^{\mathbb{R}} \to V^{\mathbb{R}}$ by the formula

$$2\Gamma(f,g) := \Delta(fg) - (f)(\Delta g) - (\Delta f)(g),$$

  which may also be expressed pointwise by

$$2\left[\Gamma(f,g)\right](x) = \frac{1}{d_x}\sum_{y\sim x}(f(y) - f(x))(g(y) - g(x)). \tag{1}$$

- The **Ricci curvature operator** denoted Γ_2 is defined by iterating the first curvature operator. Γ_2 is defined by the equation

$$2\Gamma_2(f,g) := \Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(\Delta f,g). \tag{2}$$

- A function $f \in V^{\mathbb{R}}$ is said to satisfy the curvature-dimension type inequality $CD(m,\kappa)$ for $m \in (1,\infty)$ and $\kappa \in \mathbb{R}\backslash\{0\}$ if

$$\Gamma_2(f,f) \geq \frac{1}{m}|\Delta f|^2 + \kappa\Gamma(f,f) \tag{3}$$

  at each vertex $x \in V$. If this holds for every $f \in V^{\mathbb{C}}$ then we say that G satisfies $CD(m,\kappa)$.

Connecting Ricci curvature to spectral graph theory

- A 2010 paper by Yong Lin and S.T. Yau[8] appears to be the first connection between Bakry-Émery graph curvature and spectral graph theory.

Theorem (Lin, Yau[8])

Let G be a finite connected graph with diameter D and maximum degree d. Letting λ be the smallest nonzero eigenvalue of $-\Delta$ (sometimes called Dirichlet eigenvalue), it holds

$$\lambda \geq \frac{1}{dD(\exp\{dD+1\}-1)}.$$

- They showed this by showing that general simple, connected graphs always satisfy $CD(2, \frac{1}{d}-1)$ and then leveraging the curvature inequality to bound the eigenvalue.
- Specializing to various classes of graphs as a function of their curvature parameters allows one to sharpen the estimate.

Harnack Inequalities and curvature

- 2014 saw a couple of developments connecting graph Ricci curvature to Harnack inequalities and eigenvalue estimates.
- The first we'll consider uses the techniques of Lin and Yau[8] with improvements on the techniques of Chung and Yau[3] to supply a general Harnack inequality and eigenvalue estimate.

Theorem (Chung, Yau, Lin[2])

Let G be a finite connected graph satisfying $CD(m, \kappa)$ and let $f \in V^{\mathbb{R}}$ be a harmonic eigenfunction for $-\Delta$ with nontrivial eigenvalue $\lambda > 0$. Then for each $x \in V$ it holds

$$|\nabla f|^2(x) \leq \left(\left(8 - \frac{2}{m}\right)\lambda - 4\kappa\right) \sup_{y \in V} |f|^2(y)$$

where for any $x \in V$,

$$|\nabla|^2 f(x) = \sum_{y \sim x} (f(x) - f(y))^2.$$

Harnack inequalities and curvature - continued

Theorem (Chung, Yau, Lin[2])

Let G be a finite connected graph satisfying $CD(m,\kappa)$ and $\lambda > 0$ a nontrivial eigenvalue of Δ. Then it holds

$$\lambda \geq \frac{1 + 4\kappa d D^2}{d(8 - \frac{2}{m}D^2)}.$$

In particular, for "Ricci flat" graphs satisfying $CD(m,0)$ it holds

$$\lambda \geq \frac{1}{8dD^2}$$

- The same year Man[6] specialized to the case of abelian homogeneous graphs to obtain the following.

Theorem (Man[6])

Let G be an abelian homogeneous invariant graph with generating set K consisting of k elements. Let G satisfy curvature dimension inequality $CD(m,\kappa)$ and let $f \in V^{\mathbb{R}}$ be a harmonic eigenfunction of $-\Delta$ with nontrivial eigenvalue $\lambda > 0$. Then f satisfies, at each $x \in V$,

$$\frac{1}{k}\sum_{a\in K}(f(x) - f(ax))^2 \leq \left(\left(9 - \frac{2}{m}\right)\lambda - 4\kappa - 4\right)\max_{y\in V} f(y)^2$$

- In the same paper, Man[6] proves that the eigenvalue estimate due to Chung, Lin, and Yau is strict for abelian homogeneous graphs that are Ricci flat.

Theorem (Man[6])

Let G be an abelian homogeneous invariant graph with generating set K on k elements with diameter D satisfying $CD(m, 0)$*. Let* $f \in V^{\mathbb{R}}$ *be a harmonic eigenfunction of* $-\Delta$ *with nontrivial eigenvalue* $\lambda > 0$*. Then it holds*

$$\lambda \geq \frac{1 + 4kD^2}{(9 - \frac{2}{m})kD^2}.$$

Moreover one can show that since $m > 1$ *by design of* $CD(m, \kappa)$ *in general,*

$$\lambda > \frac{1}{8kD^2}.$$

Strong Harnack Inequality

- The most recent development on this subject is work done by Chung and Yau in 2017, strengthening the results for general graphs.
- The proof is overall more sophisticated that the preceding approaches, utilizing complex analysis.

Theorem

Let G be a finite connected graph with maximum degree d satisfying $CD(m, \kappa)$, and let $f \in V^{\mathbb{R}}$ be a harmonic eigenfunction of $-\Delta$ with nontrivial eigenvalue $\lambda > 0$. Then for any pair of adjacent vertices $x, y \in V$ it holds

$$|f(x) - f(y)|^2 \leq (8\lambda + 4\kappa)d$$

Moreover if G has diameter D, it holds

$$\lambda \geq \frac{1}{8D^2 d} - \frac{\kappa}{2}$$

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