A CONTRACTION METHOD FOR BOUNDARY VALUE PROBLEMS ON MAGNETIC GRAPHS

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Let $G = (V, E)$ be a locally finite, connected, simple graph and $\Omega \subset V$ a finite subset of n vertices whose induced graph is connected. For $x, y \in V$, we write $x \sim y$ if x and y are adjacent. The boundary $\partial\Omega$ is the set

$$\partial\Omega := \{y \in V : y \notin \Omega \text{ and } \exists x \in \Omega, x \sim y\}$$

and $\overline{\Omega} := \Omega \cup \partial\Omega$. We define

$$L^2(\Omega) := \{f : \Omega \to \mathbb{C}\}$$

with inner product given by

$$\langle f, g\rangle := \sum_{x\in\Omega} f(x)\overline{g(x)}$$

for each $f, g \in L^2(\Omega)$. The *oriented edge set* of G is

$$E^{\text{or}} := \{(x, y), (y, x) \mid x, y \in V,\ x \sim y\}.$$

A *signature* on G is a map $\sigma : E^{\text{or}} \to S_p : (x, y) \mapsto \sigma_{xy}$, where S_p is the group of p-th roots of unity, satisfying

$$\sigma_{yx} = \overline{\sigma_{xy}}.$$

A pair (G, σ) is called a *magnetic graph*. If $\tau : V \to S_p$ is any function and σ any signature, the *switched* signature σ^τ is defined by

$$\sigma^\tau_{xy} = \tau(x)\sigma_{xy}\overline{\tau(v)}. \tag{1}$$

If, for two signatures σ, σ' there exists such a function τ relating them in the manner of equation (1), the two are said to be *switching equivalent*. If a signature σ is switching equivalent to the trivial signature (which associates to each oriented edge the unit element), σ is called *balanced*. Otherwise, the signature is unbalanced.

Letting σ be a fixed signature, the magnetic Laplacian operator $\Delta^\sigma : L^2(\Omega) \to L^2(\Omega)$ is defined via the equation

$$(\Delta^\sigma f)(x) = \sum_{y\sim x} (f(x) - \sigma_{xy} f(y)) \tag{2}$$

for each $f \in L^2(\Omega), x \in \Omega$. One verifies that Δ^σ is self-adjoint and positive-semidefinite. Viewing Ω as a connected subgraph of G, we denote by Δ^σ_Ω the Laplacian given by the formula in equation (2) with the summation restricted to neighbors y strictly inside Ω.

We enumerate the nonnegative eigenvalues $\mu_1, \dots, \mu_n$ in increasing order, i.e.

$$0 \le \mu_1 \le \mu_2 \le \cdots \le \mu_n.$$

Moreover, as made explicit in, e.g., [1, Equation 2.11], $\mu_1 = 0$ if and only if the signature σ is balanced.

A function $f : \Omega \times \mathbb{C} \to \mathbb{C}$ is said to be locally Lipschitz provided at each $x \in \Omega$ the function $f(x, \cdot) : \mathbb{C} \to \mathbb{C}$ is a Lipschitz function (with Lipschitz constant depending on x). For such a locally Lipschitz function $f : \Omega \times \mathbb{C} \to \mathbb{C}$ we define $\text{Lip}_\Omega(f)$ to be the maximum of the Lipschitz constants of $f(x, \cdot)$ for $x \in \Omega$.

The main boundary value problem of interest is

$$\begin{cases} \Delta^\sigma u = f(x, u) & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \end{cases}. \tag{3}$$

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for a given locally Lipschitz function f and boundary condition $g \in L^2(\partial\Omega)$.

Theorem 1. *Let (G, σ) be a magnetic graph and $\Omega \subset V$ a finite subset of n vertices whose induced graph is connected and so that the restriction of the signature to the induced subgraph is unbalanced. For any $g \in L^2(\partial\Omega)$ and any Locally Lipschitz function $f : \Omega \times \mathbb{C} \to \mathbb{C}$ satisfying $\mathrm{Lip}_\Omega(f) < \mu_n$, where μ_n is the greatest eigenvalue of Δ^σ_Ω, the problem (3) has a unique solution $u \in L^2(\overline{\Omega})$.*

Proof. The proof is inspired by the abstract monotone iteration scheme of Nieto[2], reworked as a contraction argument. Consider for $v \in L^2(\overline{\Omega})$ and $w \in L^2(\Omega)$ the auxillary problem

$$\begin{cases} (\Delta^\sigma + \lambda)v(x) = f(x, w) + \lambda w(x) & x \in \Omega \\ v(x) = g(x) & x \in \partial\Omega \end{cases}. \tag{4}$$

We claim the operator $T_\lambda : L^2(\Omega) \to L^2(\Omega) : w \mapsto v$ which sends $w \in L^2(\Omega)$ to the solution v of the corresponding problem (4) (with boundary values understood to agree with (4)) is well-defined. To see this, enumerate the vertices of $\Omega = \{x_1, \dots, x_n\}$ and $\partial\Omega = \{y_1, \dots, y_k\}$. Define the $n \times n$ matrix P by

$$P_{ij} = \begin{cases} \sigma_{x_i x_j} & x_i \sim x_j \\ 0 & \text{otherwise} \end{cases},$$

and the $n \times k$ matrix B by

$$B_{ij} = \begin{cases} \sigma_{x_i y_j} & x_i \sim y_j \\ 0 & \text{otherwise} \end{cases}.$$

so that the problem (4) may be rewritten as the matrix equation

$$-(P - (D + \lambda))v = f + \lambda w + Bg, \tag{5}$$

where D is the $n \times n$ diagonal matrix of degrees of vertices in Ω. Define the operators $L := P - D$ and $N := f(x, \cdot) + Bg$ so that the preceding becomes

$$-(L - \lambda)v = (N + \lambda)w. \tag{6}$$

Choose $\lambda_1 < 0$ with $|\lambda_1| > ||D - P||$ so that for $\lambda \leq \lambda_1$ the operator $L - \lambda$ is invertible. The operator of interest T_λ thusly has the representation

$$T_\lambda = -(L - \lambda)^{-1} \circ (N + \lambda). \tag{7}$$

whence T_λ is well defined. We claim that as a (nonlinear) operator on $L^2(\Omega)$, T_λ is a contraction if and only if $\mathrm{Lip}_\Omega(f) < \mu_n$. First note, using the preceding equation,

$$\mathrm{Lip}(T_\lambda) \leq ||(L - \lambda)^{-1}|| \left(\mathrm{Lip}_\Omega(f) + |\lambda|\right). \tag{8}$$

where $\mathrm{Lip}(T_\lambda)$ is the Lipschitz constant of T_λ as a mapping on $L^2(\Omega)$. One checks that $L = -\Delta^\sigma_\Omega$, so that for some $n \times n$ unitary matrix U, it holds

$$(L - \lambda)^{-1} = U^{-1} \begin{bmatrix} -\frac{1}{\mu_1 + \lambda} & & & \\ & -\frac{1}{\mu_2 + \lambda} & & \\ & & \ddots & \\ & & & -\frac{1}{\mu_n + \lambda} \end{bmatrix} U,$$

so that $||L - \lambda||^{-1} = \frac{1}{\mu_n + |\lambda|}$ since the greatest eigenvalue of $(L - \lambda)^{-1}$, and in turn its operator norm, is achieved when μ_i has greatest absolute value for $1 \leq i \leq n$ (recall $\lambda < 0$ and each of the eigenvalues $\mu_i > 0$ via the unbalanced condition). In turn, it holds

$$\mathrm{Lip}(T_\lambda) \leq \frac{\mathrm{Lip}_\Omega(f) + |\lambda|}{\mu_n + |\lambda|}.$$

It then follows immediately that T_λ is a contraction if and only if $\mathrm{Lip}_\Omega(f) < 1$.

To produce a solution to the main problem in equation (3), choose a starting function $u_0 \in L^2(\Omega)$ for a contractive iteration scheme; e.g., $u_0 \equiv 0$. Define

$$u_{j+1} := T_\lambda(u_j), \qquad j \geq 0.$$

Then since T_λ is a constraction, there is a limit $u \in L^2(\Omega)$ satisfying for each $x \in \Omega$, via equations (5) and (6),

$$-(P - (D + \lambda))u = f(x, u) + \lambda u + Bg,$$

or, if we extend u to the boundary via the boundary condition g, it holds

$$(D - (P + B))\, u = f(x, u)$$

at each $x \in \Omega$. One verifies that $D - (P + B) = \Delta^\sigma$ and the extension solves problem (3). □

Corollary 2. *Let (G, σ) be a magnetic graph and $\Omega \subset V$ a finite subset of vertices whose induced graph is connected and so that the restriction of the signature to the induced subgraph is unbalanced. For any $g \in L^2(\partial\Omega)$ and any $f \in L^2(\Omega)$, the Poisson problem*

$$\begin{cases} (\Delta^\sigma u)(x) = f(x) & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \end{cases}. \tag{9}$$

has a unique solution $u \in L^2(\overline{\Omega})$.

References

[1] Carsten Lange, Shiping Liu, Norbert Peyerimhoff, and Olaf Post. Frustration index and Cheeger inequalities for discrete and continuous magnetic Laplacians. *Calc. Var. Partial Differential Equations*, 54(4), 2015.

[2] Juan J. Nieto. An abstract monotone iterative technique. *Nonlinear Anal.*, 28(12), 1997.