KANTOROVICH DUALITY AND SIGNATURE BALANCE FOR GENERALIZED SIGNED GRAPHS

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Abstract. A twofold generalization of results from the theory of signed graphs to the case of signatures taking values in arbitrary groups which then act on a normed space. The first concerns the notion of signature balance, establishing a three-part characterization of the trait demonstrated by some signatures. The second is a Kantorovich-type duality between Lipschitz- and Arens-Eells-type function spaces whose norm and generator structures, respectively, incorporate the signature.

1. Introduction and Notation

1.1. **Introduction.** Signed graphs are combinatorial graphs equipped with a structure known as a signature, which is a map assigning to each edge an element of a group which then acts on some normed space (see ensuing definitions). Magnetic graphs, the particular case where the group is the unit circle in the complex plane, have appeared in various forms since at least the 1950s [1], with applications ranging from the social sciences [1] to quantum mechanics [8] and molecular modeling [5]. A similar (if more general) type of signed graph is the connection graph, again the particular case where the group consists of orthogonal matrices, which is distinguished from magnetic graphs by having a signature structure whose group acts on higher dimensional spaces [9]. Both of these types of graphs have associated Laplace operators which, while structured overall in the manner of the classical combinatorial Laplace operator for graphs [2], incorporate their respective signature structures. Much of the literature in the past few years has been dedicated to investigating relationships between the geometry of such graphs and the eigenvalues of magnetic and connection Laplacians [7, 8, 9].

Simultaneously, researchers in many areas of computational discrete mathematics (esp. graph theory, computer science) have demonstrated a renewed interest in posing and solving optimal transport problems on discrete spaces [4, 6, 11]; an interest primarily motivated by the multifaceted applicability of these problems. This paper sits at the intersection of these two research areas: signed graphs and optimal transport. The main goal is to establish a Kantorovich duality result between Lipschitz- and Arens-Eells-type function spaces using a classical approach, such as in the manner of Weaver [13], done in the case of combinatorial graphs. This duality is part of the bedrock of general optimal transport theory, both in the continuous and discrete settings [12, 13]. Our function spaces are distinguished from the classical theory in that their norm structures and generating elements, respectively, incorporate the signature structure associated with the graph.

This paper is divided into two parts, the latter of which is dedicated to proving the duality in the case where the signature is taking values in a general group which itself is acting on an arbitrary Banach space, thereby capturing both of the aforementioned examples and many more in one go. This duality, however, is subtle in the sense that it will only occur for certain types of signed graphs which are not balanced, a term used to characterize signed graphs which, put vaguely, possess signature structures which are in some sense equivalent to a trivial signature structure. The notion of signature balance has been encountered and studied many times by researchers [7, 9], and is often paired with equivalent characterizations illustrating both its significance to the theory and flexibility in appearance. This motivates the secondary goal of the paper, explored in the first part, which is to generalize the notion of balance to case of general signed graphs.

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One interesting complication that arises in generalizing the standard duality and balance results is that in moving from a space like $\mathbb{C}$ or $\mathbb{R}^n$ to a general Banach space X one loses, unsurprisingly, the self-duality properties and Hilbertian structures inherent to those finite-dimensional spaces. It is because of this that both of the principal results on balance and duality are paired with ‘dual’ versions. In particular, each signature whose group action is on a normed space X yields a dual action on X^*, its continuous dual; balance characterizations are then expressed separately for the original signature’s group action and its dual action. A similar theme shows up in the duality results, with one posed for Arens-Eells functions taking values in X and a second posed for functions taking values in X^*.

We punctuate the results with an appendix in the last section which contains mostly classical results known to many experts but which may be new to the reader less familiar with functional analysis on normed spaces. Some results in the appendix are just one or two steps beyond the general results seen in reference texts on Banach spaces and operator theory and are included for completeness.

1.2. **Notations and definitions.** Let $G = (V, E)$ be a connected, simple graph (that is, finite vertex set, no loops or multiple edges) whose vertices we label u, v, w as needed. Γ is a group whose elements we denote g, h; e is its identity element. We will use the big pi notation for general group element products, i.e.

$$\prod_{i=1}^{n} g_i := g_1 \cdot g_2 \cdots g_{n-1} \cdot g_n, \quad \prod_{i=n}^{1} g_i := g_n \cdot g_{n-1} \cdots g_2 \cdot g_1, , \quad g_i \in \Gamma,\ 1 \leq i \leq n.$$

By ordering the index set clearly, we do not make any commutativity assumptions.

The *oriented edges* of G are given by

$$E^{\mathrm{or}}(G) := \{(u, v), (v, u) \mid u, v \in V, u \sim v\}$$

and a *signature* is a map $\sigma : E^{\mathrm{or}}(G) \to \Gamma$ satisfying $\sigma_{vu} = (\sigma_{uv})^{-1}$. The *trivial signature* is the one which assigns the identity element to every oriented edge.

Let $(X, ||\cdot||_X)$ be a normed space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, which we take to be complete as necessary. X^* will denote its dual space of continuous linear functionals, and we will use the angle bracket notation $\langle \phi, x\rangle$ to denote the value of a functional $\phi \in X^*$ on a vector $x \in X$. If two normed spaces X, Y are isometrically isomorphic then we write $X \equiv Y$. V^X will denote the vector space of all functions $f : V \to X$.

A *left action* α of Γ on a set A is an association $g \mapsto \alpha(g)$ from the group to a mapping $A \to A$ such that

$$\begin{array}{ll} \text{(identity)} & \alpha(e)x = x \text{ for each } x \in A \\ \text{(compatibility)} & \alpha(gh)x = \alpha(g)(\alpha(h)x) \text{ for each } g, h \in G \text{ and } x \in A. \end{array}$$

An *anticompatible left action* β of Γ on a set A is an association $g \mapsto \beta(g)$ from the group to a mapping $A \to A$ such that

$$\begin{array}{ll} \text{(identity)} & \beta(e)x = x \text{ for each } x \in A \\ \text{(anticompatibility)} & \beta(gh)x = \beta(h)(\beta(g)x) \text{ for each } g, h \in G \text{ and } x \in A. \end{array}$$

If α is a left action of Γ on X, we define the *dual action* α^* of Γ on X^* via

$$\langle \alpha^*(g)\phi, x\rangle = \langle \phi, \alpha(g)x\rangle \tag{1}$$

Note that α^* is an anticompatible left action of Γ on X^*.

2. On balance

To motivate the first key result, we ask the reader to consider the following two propositions from graph theory concerning certain specific examples of signed graphs. Both settings contain subfamilies of graphs which are often called *balanced*, a property revealed by many different characterizations which the propositions at issue serve to identify. In broad strokes, balanced signed graphs have the property that the product of signature values along oriented cycles always yield the identity element of the group, as well as the property that the signature structure is somehow equivalent to the trivial signature, in a manner made precise by the following two propositions.

The first proposition concerns what are known as *magnetic graphs*. A magnetic graph is a pair (H, ρ) where H is a simple graph and ρ is a signature taking values in the group $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. In the past, signature structures on these types of graphs have served to discretely model magnetic fields or quantum mechanical systems [8]. In this setting, the group S^1 acts by scalar multiplication on the codomain of the functions of interest, $\mathbb{C}$. Moreover, the duality structure is in the form of an inner product. If $\tau : V \to \mathbf{S}^1$ is some function, then we may produce the τ-switched signature denoted ρ^τ via

$$\rho^\tau_{uv} := \tau(u)\rho_{uv}\tau(v)^{-1}. \tag{2}$$

Two distinct signatures related in this manner by some switching function are said to be *switching equivalent*.

Proposition 2.1. *The following are equivalent:*

(i) A magnetic graph (H, ρ) is balanced under the action of S^1 on $\mathbb{C}$.

(ii) For every oriented cycle expressed as a list of incident oriented edges in the form

$$((u_0, u_1), (u_1, u_2), \ldots, (u_{n-1}, u_n = u_0)),$$

it holds $\prod_{i=0}^{n-1} \rho_{u_i u_{i+1}} = 1$.

(iii) ρ is switching equivalent to the trivial signature.

The relationship between balance and switching equivalence is noted by Lange, Liu, Peyerimhoff and Post [7, Proposition 3.2].

The second type of graph structure with which we are concerned is *connection graphs*, that is, pairs (F, ω) where F is a simple graph and $\omega : E^{\mathrm{or}}(F) \to O_n(\mathbb{F})$ is a signature on F taking values in the group $O_n(\mathbb{F})$ of orthogonal matrices with entries in the field $\mathbb{F}$. In this setting, the normed space of interest is $\mathbb{F}^n$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. These types of graphs are of interest mathematically as a generalization of magnetic graphs, but also for their use in applied topics [9]. The action here of $O_n(\mathbb{F})$ on $\mathbb{F}^n$ is matrix multiplication.

Proposition 2.2. *The following are equivalent:*

(i) A connection graph (F, ω) is balanced under the action of $O_n(\mathbb{F})$ on $\mathbb{F}^n$.

(ii) For every oriented cycle expressed as a list of incident oriented edges in the form

$$((u_0, u_1), (u_1, u_2), \ldots, (u_{n-1}, u_n = u_0)),$$

it holds $\prod_{i=0}^{n-1} \omega_{u_i u_{i+1}} = Id_n$, the $n \times n$ identity matrix.

(iii) ω is switching equivalent to the trivial signature, in the sense that there exists $T : V \to O_n(\mathbb{F})$ such that

$$T(u)\omega_{uv}T(v)^{-1} = Id_n$$

for each $u \sim v$.

In both of the preceding examples, the spaces on which the groups were acting ($\mathbb{F}, \mathbb{F}^n$ resp.) have Hilbertian, and hence self-dual, structures. A general Banach space, of course, does not. This complication only substantively affects the duality results in the subsequent section of this paper, but motivates the development of not one but two characterizations of signature balance: one for the action of the signature on the Banach space, and a second for the dual action of the signature on the dual to the Banach space. The second result is phrased more generally than it is motivated, in terms of an anti-compatible left action on an arbitrary normed space.

Theorem 2.3 (Signature equivalence (a))**.** *Let $G = (V, E)$ be a connected simple graph, σ a signature on G taking values in a group Γ, X a normed space, and α an action of of Γ on X. Then the following are equivalent:*

(i) There exists a function $f : V \to X$, not identically 0, such that for every oriented edge $(u, v) \in E^{or}(G)$, it holds
$$f(u) = \alpha(\sigma_{uv})f(v).$$

(ii) There exists a function $f : V \to X$, not identically 0, such that for each $u \in V$ and each directed cycle of the form
$$((u = u_0, u_1), (u_1, u_2), \ldots, (u_{\ell-2}, u_{\ell-1}), (u_{\ell-1}, u_\ell = u)) \subset E^{or}(G)$$
it holds
$$\alpha\left(\prod_{i=0}^{\ell-1} \sigma_{u_iu_{i+1}}\right) f(u) = f(u),$$

(iii) There exists a nonzero element $x \in X$ and some $u_0 \in V$ such that for each directed cycle of the form
$$((u_0, u_1), (u_1, u_2), \ldots, (u_{n-2}, u_{n-1}), (u_{n-1}, u_n = u_0)) \subset E^{or}(G)$$
it holds
$$\alpha\left(\prod_{i=0}^{n-1} \sigma_{u_iu_{i+1}}\right) x = x.$$

Proof. First let us assume condition (i) holds, that is, let $f : V \to X$ be some nonzero function satisfying $f(u) = \alpha(\sigma_{uv})f(v)$ for every oriented edge $(u, v) \in E^{\text{or}}(G)$. We claim that this function f satisfies (ii). Let $u \in V$ be fixed and suppose
$$((u = u_0, u_1), (u_1, u_2), \ldots, (u_{n-2}, u_{n-1}), (u_{n-1}, u_n = u)) \subset E^{\text{or}}(G)$$
is a directed cycle originating and terminating at u. Then,
$$\begin{aligned}\alpha\left(\prod_{i=0}^{n-1} \sigma_{u_iu_{i+1}}\right) f(u) &= \left(\prod_{i=0}^{n-1} \alpha(\sigma_{u_iu_{i+1}})\right) f(u)\\ &= \alpha(\sigma_{u_0u_1}) \cdot \alpha(\sigma_{u_1u_2}) \cdots \alpha(\sigma_{u_{n-1}u_n})f(u_n)\\ &= \alpha(\sigma_{u_0u_1}) \cdot \alpha(\sigma_{u_1u_2}) \cdots \alpha(\sigma_{u_{n-2}u_{n-1}})f(u_{n-1})\\ \cdots &= \alpha(\sigma_{u_0u_1})f(u_1) = f(u)\end{aligned}$$
as desired.

That (ii) implies (iii) is clear, i.e. choose any nonzero value $f(u)$ of the function in (ii) and condition (iii) holds.

It remains to prove that condition (iii) implies condition (i). To this end, let $u_0 \in V, x \in X$ satisfy (iii). We need to supply a function f as in (i); to do this, define $f(u_0) = x$. Then for any vertex $u \in V$, let
$$((u = v_0, v_1), (v_1, v_2), ..., (v_{n-2}, v_{n-1}), (v_{n-1}, v_n = u_0)) \subset E^{\text{or}}(G)$$
be a path originating at u and terminating at u_0. Temporarily put
$$f_1(u) = \alpha\left(\prod_{i=0}^{n-1} \sigma_{v_iv_{i+1}}\right) x. \tag{3}$$

Now let
$$((u = v'_0, v'_1), (v'_1, v'_2), ..., (v'_{m-2}, v'_{m-1}), (v'_{m-1}, v'_m = u_0)) \subset E^{\text{or}}(G)$$

be a separate path between u and u_0 and temporarily set

$$f_2(u) = \alpha\left(\prod_{i=0}^{m-1} \sigma_{v'_i v'_{i+1}}\right) x.$$

Notice that

$$((v'_m = u_0, v'_{m-1}), \dots, (v'_2, v'_1), (v'_1, v'_0 = u), (v_0, v_1), (v_1, v_2), \dots, (v_{n-1}, v_n = u_0))$$

is an oriented cycle originating and terminating at u_0. Via (iii) it holds

$$\alpha\left(\prod_{j=m}^{1} \sigma_{v'_j v'_{j-1}}\right) \alpha\left(\prod_{i=0}^{n-1} \sigma_{v_i v_{i+1}}\right) x = x.$$

In turn,

$$\alpha\left(\prod_{j=m-1}^{1} \sigma_{v'_j v'_{j-1}}\right) \alpha\left(\prod_{i=0}^{n-1} \sigma_{v_i v_{i+1}}\right) x = \alpha(\sigma_{v'_{m-1} v'_m}) x$$

$$\alpha\left(\prod_{j=m-2}^{1} \sigma_{v'_j v'_{j-1}}\right) \alpha\left(\prod_{i=0}^{n-1} \sigma_{v_i v_{i+1}}\right) x = \alpha(\sigma_{v'_{m-2} v'_{m-1}})\alpha(\sigma_{v'_{m-1} v'_m}) x$$

$$\alpha\left(\prod_{i=0}^{n-1} \sigma_{v_i v_{i+1}}\right) x = \alpha(\sigma_{v'_0 v'_1})\alpha(\sigma_{v'_1 v'_2}) \cdots \alpha(\sigma_{v'_{m-2} v'_{m-1}})\alpha(\sigma_{v'_{m-1} v'_m}) x.$$

That is, $f_1(u) = f_2(u)$ ensuring that our path-based construction is well-defined. Extend f to every $u \in V$ as in equation (3). Suppose $(u, v) \in E^{(or)}(G)$ is a fixed oriented edge, and let

$$((v_0 = v, v_1), (v_1, v_2), ..., (v_{k-2}, v_{k-1}), (v_{k-1}, v_k = u_0)) \subset E^{\mathrm{or}}(G)$$

be a path connecting v to u_0. Then it holds

$$f(u) = \alpha(\sigma_{uv}) \left(\prod_{i=0}^{k-1} \alpha(\sigma_{v_i v_{i+1}})\right) x = \alpha(\sigma_{uv}) f(v)$$

as claimed. □

What follows is the aforementioned 'dual' version to the preceding theorem, whose proof is symmetrically identical to that of Theorem 2.3, with care taken to reverse the order of certain products to accommodate the anticompatibility of the action in question.

Theorem 2.4 (Signature equivalence (b))**.** *Let $G = (V, E)$ be a connected simple graph, σ a signature on G taking values in a group Γ, X a normed space, and β an anticompatible action of of Γ on X. Then the following are equivalent:*

(i) There exists a function $f : V \to X$, not identically 0, such that for every oriented edge $(u, v) \in E^{or}(G)$, it holds

$$f(u) = \beta(\sigma_{uv}) f(v).$$

(ii) There exists a function $f : V \to X$, not identically 0, such that for each $u \in V$ and each directed cycle of the form

$$((u = u_0, u_1), (u_1, u_2), ..., (u_{\ell-2}, u_{\ell-1}), (u_{\ell-1}, u_\ell = u)) \subset E^{or}(G)$$

it holds

$$\beta\left(\prod_{i=0}^{\ell-1} \sigma_{u_{i+1} u_i}\right) f(u) = f(u),$$

(iii) There exists a nonzero element $x \in X$ and some $u_0 \in V$ such that for each directed cycle of the form
$$((u_0, u_1), (u_1, u_2), ..., (u_{n-2}, u_{n-1}), (u_{n-1}, u_n = u_0)) \subset E^{or}(G)$$
it holds
$$\beta\left(\prod_{i=0}^{n-1} \sigma_{u_{i+1}u_i}\right) x = x.$$

Let us introduce some terminology we will use to refer to these two types of signatures/left actions.

(i) σ is said to be *balanced under the action* α *on* X if any of the conditions in Theorem 2.3 is satisfied.

(ii) σ is said to be $*$*-balanced under the action* β *on* X if any of the conditions in Theorem 2.4 is satisfied.

3. On duality

In this section, we are interested in proving that two spaces, one Lipschitz-type and another Arens-Eells type, are related by duality when a signed graph satisfies various balance conditions as needed. The main argument will require several preliminary results which will constitute the first part of this section. The first lemma is somewhat independent from the subsequent ones and is mainly supplied for its proof which will be helpful in proving the third lemma. Throughout this section, the normed space X we assume to be complete. Also, we will have several occasions to use classical results whose proofs we have placed in an appendix.

Let us first introduce two useful normed spaces. We define $\ell_1(V; X) := (V^X, ||\cdot||_{\ell_1(V;X)})$, where
$$||f||_{\ell_1(V;X)} = \sum_{u \in V} ||f(u)||_X .$$
Similarly we define $\ell_\infty(V; X) := (V^X, ||\cdot||_{\ell_\infty(V;X)})$, where
$$||f||_{\ell_\infty(V;X)} = \max_{u \in V} ||f(u)||_X .$$

Lemma 3.1. *We have the identification* $(\ell_1(V; X))^* \equiv \ell_\infty(V; X^*)$.

Proof. This lemma can be proved as a straightforward consequence of Lemma 4.1, proved in the appendix. For the purposes of a subsequent argument, we will supply the particular mappings T_1, T_2 which play the same role as those in the proof of Lemma 4.1, but contextualized to these spaces of interest. The idea comes from [13, Thm. 2.2.2]. Begin by defining $T_1 : \ell_\infty(V; X^*) \to (\ell_1(V; X))^*$ via the formula
$$\langle T_1 p, f\rangle = \sum_{u \in V} \langle p(u), f(u)\rangle$$
for any $p \in \ell_\infty(V; X^*)$ and $f \in \ell_1(V; X)$. The inverse mapping $T_2 : (\ell_1(V; X))^* \to \ell_\infty(V; X^*)$ is given by the formula
$$\langle (T_2\phi)(u), x\rangle = \langle \phi, x\delta_u\rangle$$
for each $\phi \in (\ell_1(V; X))^*$, $u \in V$ and $x \in X$. As an exercise the reader may verify the straightforward estimates $||T_1|| \le 1$ and $||T_2|| \le 1$, as well as $T_1T_2 = \mathrm{Id}$, $T_2T_1 = \mathrm{Id}$ to complete the proof. $\square$

Definition 3.2. *The signed Lipschitz space is given by*
$$\mathrm{Lip}^\sigma(G; X) := \{f : V \to X \mid \exists C \ge 0 \ s.t. ||f(u) - \alpha(\sigma_{uv})f(v)||_X \le C \ \forall u \sim v\}.$$
equipped with the semi-norm
$$||f||_{\mathrm{Lip}^\sigma(G;X)} := \max_{u \sim v} ||f(u) - \alpha(\sigma_{uv})f(v)||_X .$$

Definition 3.3. *Letting $a \in X$ and $u, v \in V$ be any adjacent vertices, we can define $am^\sigma_{uv} : V \to X$ via*

$$am^\sigma_{uv}(w) = \begin{cases} a & \text{if } w = u \\ -\alpha(\sigma_{uv})a & \text{if } w = v \\ 0 & \text{otherwise} \end{cases}, \quad \forall w \in V.$$

along with the signed Arens-Eells space

$$Æ^\sigma(G; X) := \{am^\sigma_{uv} \mid u, v \in V, u \sim v, a \in X\}$$

equipped with the semi-norm defined by

$$||m||_{Æ^\sigma(G;X)} = \inf\left\{\sum_{i=1}^n ||a_i||_X \mid m = \sum_{i=1}^n a_i m^\sigma_{u_i v_i}, \quad u_i \sim v_i, a_i \in X,\ 1 \le i \le n\right\}.$$

It is clear that both of our defined norm structures above form semi-norms; that is, they are homogeneous and subadditive. While $Æ^\sigma(G; X)$ will be a true normed space in general, $\mathrm{Lip}^\sigma(G; X)$ will be a normed space if σ is *not* balanced under the action of α on X. Interestingly and symmetrically, while $\mathrm{Lip}^\sigma(G; X) = V^X$ as vector spaces, $Æ^\sigma(G; X)$ will coincide with V^X when σ is *not* $*$-balanced under the action α^* on X^*.

Lemma 3.4. *$Æ^\sigma(G; X)$ is a normed space, and if σ is not balanced under the action of α on X then $\mathrm{Lip}^\sigma(G; X)$ is also a normed space.*

Proof. All that remains to be proved is that the norms on $\mathrm{Lip}^\sigma(G; X)$ and $Æ^\sigma(G; X)$ as previously defined are definite. Suppose $m \in Æ^\sigma(G; X)$ satisfies $||m||_{Æ^\sigma(G;X)} = 0$. For positive integer k, find some finite linear combination of atoms $\sum_i a_i^k m^\sigma_{u_i^k v_i^k}$ for which

$$m = \sum_{i=1}^{n_k} a_i^k m^\sigma_{u_i^k v_i^k}, \quad \sum_{i=1}^{n_k} ||a_i^k||_X < \frac{1}{k}.$$

Then,

$$||m||_{\ell_1(V;X)} = \left|\left|\sum_{i=1}^{n_k} a_i^k m^\sigma_{u_i^k v_i^k}\right|\right|_{\ell_1(V;X)} \le \sum_{i=1}^{n_k} \left|\left|a_i^k m^\sigma_{u_i^k v_i^k}\right|\right|_{\ell_1(V;X)} \le 2\sum_{i=1}^{n_k} ||a_i^k||_X < \frac{2}{k} \to 0 \text{ as } k \to \infty.$$

From the definiteness of the $||\cdot||_{\ell_1(V;X)}$ norm, the claim is verified. Now suppose $f \in \mathrm{Lip}^\sigma(G; X)$ satisfies $||f||_{\mathrm{Lip}^\sigma(G;X)} = 0$. This means that for every edge $(u, v) \in E^{\mathrm{or}}(G)$, it holds $f(u) = \alpha(\sigma_{uv})f(v)$, so either $f = 0$ or σ is balanced under the action α on X by the characterization in Theorem 2.3 (condition 2), a contradiction. $\square$

Via a similar argument utilizing the signature equivalence in Theorem 2.4, we can get a dual version of the preceding lemma.

Lemma 3.5. *$Æ^\sigma(G; X^*)$ is a normed space, and if σ is not $*$-balanced under the action of α^* on X^* then $\mathrm{Lip}^\sigma(G; X^*)$ is also a normed space.*

Lemma 3.6. $\mathrm{Lip}^\sigma(G; X^*) = V^{X^*}$ *as vector spaces, and if σ is not balanced under the action α on X, then the equality $Æ^\sigma(G; X^*) = V^{X^*}$ holds as well.*

Proof. The first equality is clear by the simplicity of G; for the latter, suppose $Æ^\sigma(G; X^*)$ is a proper subspace of $\ell_\infty(V; X^*)$ (as vector spaces rather than Banach spaces). Then the image of this space under the mapping T_1 as supplied in the proof of Lemma 3.1, would be a proper subspace of $\ell_1(V; X)^*$. Recall that for each $\psi \in X^*$, oriented edge (u, v), and $f \in \ell_1(V; X)$ we have

$$\langle T_1(\psi m^\sigma_{uv}), f\rangle = \langle \psi, f(u) - \alpha(\sigma_{uv})f(v)\rangle.$$

If we formally define $\psi M_{uv}^{\sigma} \in \ell_1(V;X)^*$ to be the functional obtained by $T_1(\psi m_{uv}^{\sigma})$, then the image of $Æ^{\sigma}(G;X^*)$ under T_1 may be expressed

$$\widetilde{Æ} := \mathrm{Im}_{T_1}\left[Æ^{\sigma}(G;X^*)\right] = \{\psi M_{uv}^{\sigma} : \psi \in X^*,\ u \sim v\}, \quad ,$$

with

$$||M||_{\widetilde{Æ}} := \inf\left\{\sum_{i=1}^{n} ||\psi||_{X^*},\ M = \sum_{i=1} \psi_i M_{u_i v_i}^{\sigma}, u_i \sim v_i\right\}$$

Take as proved that this space is a proper weak-$*$ closed subspace in $\ell_1(V,X)^*$. Then by Lemma 4.2 we can find a a *nonzero* evaluation functional $e_f \in \ell_1(V;X)^{**}$, that is, an element of the canonical embedding of $\ell_1(V;X)$ into its double-dual corresponding to some nonzero function $f \in \ell_1(V;X)$, which is identically 0 on every element of $\widetilde{Æ}$, i.e. for every $\psi \in X^*$ and $u \sim v$, it holds

$$\langle e_f, \psi M_{uv}^{\sigma}\rangle = 0 \iff \langle \psi, f(u) - \alpha(\sigma_{uv})f(v)\rangle = 0 \iff f(u) - \alpha(\sigma_{uv})f(v) = 0$$

that is, σ is balanced under the action α on X via Theorem 2.3, contradicting the hypothesis.

Returning now to the unproven claim that $\widetilde{Æ}$ is weak-$*$ closed, we wish to invoke the Theorem 4.6. To do so we must first prove that $\widetilde{Æ}$ is both norm closed and may be represented as a finite sum of subspaces, each of which is itself weak-$*$ closed. Attacking the first point, fix a set $\mathcal{D} \subset E^{\mathrm{or}}(G)$ with the property that $(u,v) \in \mathcal{D} \iff (v,u) \notin \mathcal{D}$; this is a set of formal oriented edges containing one representative for each combinatorial edge. Given $N \in \widetilde{Æ}$, and $\epsilon > 0$, we can find a family of functionals $\{\psi_{(u,v)}\}_{(u,v)\in\mathcal{D}}$, some of which may be zero, for which

$$N = \sum_{(u,v)\in\mathcal{D}} \psi_{(u,v)} M_{uv}^{\sigma}, \qquad \text{and} \sum_{(u,v)\in\mathcal{D}} \left|\left|\psi_{(u,v)}\right|\right|_{X^*} \leq ||N||_{\widetilde{Æ}} + \epsilon.$$

So to prove that $\widetilde{Æ}$ is norm closed, via [10, Thm. 1.3.9] it suffices to prove that every absolutely convergent series in this space converges. To this end, suppose we have a sequence $(M^n)_{n\in\mathbb{N}} \subset \widetilde{Æ}$ for which $\sum_{n=1}^{\infty} ||M^n|| < \infty$. For every n we can find a family of functionals $\{\psi_{(u,v)}^n\}_{(u,v)\in\mathcal{D}}$ for which

$$M^n = \sum_{(u,v)\in\mathcal{D}} \psi_{(u,v)}^n M_{uv}^{\sigma}, \qquad \text{and} \sum_{(u,v)\in\mathcal{D}} \left|\left|\psi_{(u,v)}^n\right|\right|_{X^*} \leq ||M^n||_{\widetilde{Æ}} + \frac{1}{2^n}.$$

Then it follows that for each $(u,v) \in \mathcal{D}$,

$$\sum_{n=1}^{\infty} \left|\left|\psi_{(u,v)}^n\right|\right|_{X^*} \leq \sum_{n=1}^{\infty}\left(||M^n||_{\widetilde{Æ}} + \frac{1}{2^n}\right) < \infty.$$

Since X is complete and the series $\sum_{n=1}^{\infty} \psi_{(u,v)}^n$ is absolutely summable in X^*, it has a limit $\psi_{(u,v)}$. We claim that

$$\sum_{n=1}^{\infty} M^n = \sum_{(u,v)\in\mathcal{D}} \psi_{(u,v)} M_{uv}^{\sigma}.$$

We test their difference at the partial sum level:

$$\begin{aligned}\left|\left|\sum_{n=1}^{N} M^n - \sum_{(u,v)\in\mathcal{D}} \psi_{(u,v)} M_{uv}^{\sigma}\right|\right|_{\widetilde{Æ}} &= \left|\left|\sum_{(u,v)\in\mathcal{D}}\left(\sum_{n=1}^{N} \psi_{(u,v)}^n - \psi_{(u,v)}\right) M_{uv}^{\sigma}\right|\right|_{\widetilde{Æ}} \\ &\leq \sum_{(u,v)\in\mathcal{D}} \left|\left|\left(\sum_{n=1}^{N} \psi_{(u,v)}^n - \psi_{(u,v)}\right) M_{uv}^{\sigma}\right|\right|_{\widetilde{Æ}} \\ &\leq \sum_{(u,v)\in\mathcal{D}} \left|\left|\left(\sum_{n=1}^{N} \psi_{(u,v)}^n - \psi_{(u,v)}\right)\right|\right|_{X^*} < \epsilon \text{ for } N \text{ large,}\end{aligned}$$

since $\psi^n_{(u,v)} \to \psi_{(u,v)}$ in X^*. We conclude that $\widetilde{Æ}$ is indeed closed in the norm $||\cdot||_{\widetilde{Æ}}$. With the first point addressed, take the following decomposition:

$$\widetilde{Æ} = \sum_{(u,v)\in\mathcal{D}} \widetilde{A}_{(u,v)}, \text{ where } \widetilde{A}_{(u,v)} := \left\{\psi M^\sigma_{uv} \mid \psi \in X^*\right\}$$

It remains to be shown that these spaces are weak-$*$ closed individually. To do this, we will need to use the following evaluation operator, based on T_2 as in Lemma 3.1. Letting $w_0 \in V$ be fixed, define $T_2^{w_0} : \ell_2(V;X)^* \to X^*$ via $T_2^{w_0}(\Psi) = (T_2\Psi)(w_0)$ for each $w \in V$ and $\Psi \in \ell_1(V;X)^*$. It can easily be shown that this operator is weak-$*$ to weak-$*$ continuous by a net argument; specifically, convergence of the image of a net in X^* is immediately implied by net convergence in the domain by the definition of T_2. Now take a net $(\Psi_\lambda)_{\lambda\in\Lambda} \subset \widetilde{A}_{(u,v)}$ converging weak-$*$ to some $\Psi \in \ell_1(V;X)^*$. We want to produce some $\psi \in X^*$ for which $\Psi = \psi M^\sigma_{uv}$. The idea is to reveal a limit of this net under the mappings T^2_u, T^2_v. Since $(\Psi_\lambda)_{\lambda\in\Lambda} \subset \widetilde{A}_{(u,v)}$, we can find a net $\psi_\lambda \in X^*$ for which $\Psi_\lambda = \psi_\lambda M^\sigma_{uv}$ for each $\lambda \in \Lambda$. Notice that for any $\lambda \in \Lambda, x \in X$ we have

$$\left\langle \left(T^2_u \Psi_\lambda\right), x\right\rangle = \langle (T_2\psi_\lambda M^\sigma_{uv})(u), x\rangle = \langle \psi_\lambda M^\sigma_{uv}, x\delta_u\rangle = \langle \psi_\lambda, x\rangle.$$

That is, $(T^u_2)(\Psi_\lambda) = \psi_\lambda$. Since T^u_2 is weak-$*$ continuous, we can define ψ to be the weak-$*$ limit of $(T^u_2)(\Psi_\lambda)$ in X^*, that is, $T^u_2(\Psi)$. Using a similar argument, one can prove that $(T^v_2)(\Psi_\lambda) = -\alpha^*(\sigma_{uv})\psi_\lambda$, and in turn, $\lim(T^u_2)(\Psi_\lambda) = -\alpha^*(\sigma_{uv})\psi$. Additionally it follows readily that $T^w_2(\Psi_\lambda) = 0$ for any other vertex $w \neq u, v$. This is all to say that $T_2\Psi = \psi m^\sigma_{uv} \in \ell_\infty(V;X^*)$. One can check that the inverse image of this function under the mapping T_1, *c.f.* Lemma 3.1, is $\psi M^\sigma_{uv} \in \widetilde{A}_{(u,v)}$, proving that $\widetilde{A}_{(u,v)}$ is weak-$*$ closed. Having decomposed the norm-closed space $\widetilde{Æ}$ into the sum of finitely many weak-$*$ closed pieces, we can invoke the Theorem 4.6 and conclude as claimed earlier that $\widetilde{Æ}$ is weak-$*$ closed. $\square$

Lemma 3.7. $\mathrm{Lip}^\sigma(G;X) = V^X$ *as vector spaces, and if* σ *is not* $*$*-balanced under the action* α^* *on* X^**, then the equality* $Æ^\sigma(G;X) = V^X$ *holds as well.*

Proof. That $\mathrm{Lip}^\sigma(G;X) = V^X$ is follows from our requirement that G be simple. Suppose σ is not $*$-balanced under the action α^* on X^* and that $Æ^\sigma(G;X) = V^X$ is a proper, closed subspace of $\ell_1(V;X)$ (to see that the space is closed, use an argument similarly identical to that presented in the preceding proof). We may then obtain a functional $\Psi \in (\ell_1(V;X))^*$ which is not identically 0 but which vanishes on $Æ^\sigma(G;X)$. As we showed in the proof of Lemma 3.1, we may then obtain a functional-valued function $T_2\Psi \in \ell_\infty(V;X^*)$ which is not identically 0 satisfying $\langle T_2\Psi(u), x\rangle = \langle \Psi, x\delta_u\rangle$. Then the following holds for any $x \in X$ and any adjacent $u, v \in V$:

$$\begin{aligned}\langle (T_2\Psi)(u) - \alpha^*(\sigma_{uv})(T_2\Psi)(v), x\rangle &= \langle (T_2\Psi)(u), x\rangle - \langle (T_2\Psi)(v), \alpha(\sigma_{uv})x\rangle \\ &= \langle \Psi, x\delta u\rangle - \langle \Psi, \alpha(\sigma_{uv})x\delta_v\rangle = \langle \Psi, m^\sigma_{uv}\rangle = 0\end{aligned}$$

whence $(T_2\Psi)(u) - \alpha^*(\sigma_{uv})(T_2\Psi)(v) = 0$ for each $u \sim v$, implying $T_2\Psi$ satisfies the condition (2) in Theorem 2.4 contrary to our unbalanced assumption on σ. $\square$

We now prove the two central results of this section.

Theorem 3.8. *If* σ *is not* $*$*-balanced under the action* α^* *on* X^**, then we have the identification*

$$Æ^\sigma(G;X)^* \equiv \mathrm{Lip}^\sigma(G;X^*).$$

Proof. Note first that as a consequence of the balance assumption, via Lemmas 3.5 and 3.7, $\mathrm{Lip}^\sigma(G;X^*)$ is a normed space and $Æ^\sigma(G;X) = V^X$ as vector spaces. We will proceed as in the proof of Lemma 3.1, but supply more detail on the estimates since the norms in these spaces are different from those handled in the previous duality lemma. $T_1 : \mathrm{Lip}^\sigma(G;X^*) \to Æ^\sigma(G;X)^*$ is given by the formula

$$\langle T_1\Psi, m\rangle := \sum_{u\in V} \langle \Psi(u), m(u)\rangle.$$

For $m \in Æ^\sigma(G;X)$ fixed, let $m = \sum_{i=1}^n a_i m_{u_i v_i}^\sigma$ where $a_i \in X$ and $u_i \sim v_i$ for each $1 \leq i \leq n$. Then it holds

$$\begin{aligned}|\langle T_1\Psi, m\rangle| = \big|\sum_{u\in V}\langle\Psi(u), m(u)\rangle\big| &\leq \sum_{i=1}^n \big|\langle\Psi(u_i), a_i\rangle - \langle\Psi(v_i), \alpha(\sigma_{u_iv_i})a_i\rangle\big| \\ &= \sum_{i=1}^n \big|\langle\Psi(u) - \alpha^*(\sigma_{u_iv_i})\Psi(v_i), a_i\rangle\big| \leq ||\Psi||_{\mathrm{Lip}^\sigma(G;X^*)}\sum_{i=1}^n ||a_i||_X\end{aligned}$$

After taking an inf over the families $\{a_i\}$ which represent m, and a sup over all m with norm at most one, one obtains $||T_1|| \leq 1$. Next, $T_2 : Æ^\sigma(G;X)^* \to \mathrm{Lip}^\sigma(G;X^*)$ is given implicitly by the formula

$$\langle T_2\phi(u), x\rangle = \langle\phi, x\delta_u\rangle.$$

Notice that $x\delta_u \in Æ^\sigma(G;X)$ since σ is unbalanced. Once again we may estimate for any $\phi \in Æ^\sigma(G;X)^*$, $x \in X$ and u, v adjacent:

$$\begin{aligned}\big|\langle T_2\phi(u) - \alpha^*(\sigma_{uv})T_2\phi(v), x\rangle\big| &= \big|\langle\phi, x\delta_u\rangle - \langle\phi, \alpha(\sigma_{uv})x\delta_v\rangle\big| \\ &= |\langle\phi, xm_{uv}^\sigma\rangle| \leq ||\phi||_{Æ^\sigma(G;X)^*}\,||xm_{uv}^\sigma||_{Æ^\sigma(G;X)} \\ &\leq ||\phi||_{Æ^\sigma(G;X)^*}\,||x||_X\,.\end{aligned}$$

First taking a sup over all $||x||_X \leq 1$, then a max over all vertices $u \sim v$ will yield $||T_2|| \leq 1$. The reader is invited to verify that $T_1T_2 = \mathrm{Id}$, and $T_2T_1 = \mathrm{Id}$ to complete the proof. □

Theorem 3.9. *If σ is not balanced under the action α on X, then we have the identification*

$$Æ^\sigma(G;X^*) \equiv \mathrm{Lip}^\sigma(G;X)^*.$$

The proof of this claim follows the preceding argument *mutatis mutandis*. The hypothesis that σ is not balanced under the action α ensures that $\mathrm{Lip}^\sigma(G;X)^*$ is a well-defined normed space, via Lemma 3.4, and that $Æ^\sigma(G;X^*) = V^{X^*}$ via Lemma 3.6.

As a final remark, notice that both of the preceding duality results hold only when the signature is doubly unbalanced.

4. Appendix: Functional Analysis

This appendix will be used to write down various results used in the preceding work which are generally known to experts but may elude the casual reader. In the first lemma we write down a classical duality result, the second a useful hyperplane separation lemma, and the three which follow lead up to the main result of interest at the end. First we attend to some preliminary definitions. Throughout, X is a normed space which may or may not be complete as the theorems require.

Recall that if X is a normed space, $X^\#$ is its *algebraic dual*; that is, the linear space consisting of all (not necessarily continuous) linear functionals. Also recall the definitions of the *annihilator and pre-annihilator subspaces* for subsets A, B of a normed space X and its dual X^*, resp.:

$$\begin{aligned}A^\perp &:= \{x^* \in X^* : \langle x^*, x\rangle = 0\ \forall x \in A\} \\ {}^\perp B &:= \{x \in X : \langle x^*, x\rangle = 0\ \forall x^* \in B\}.\end{aligned}$$

Letting $X_1, X_2, \ldots X_n$ be normed spaces, we will utilize in various forms the direct product of these spaces, which we will norm in two different manners. First define the general product space

$$X_1 \oplus_p X_2 \oplus_p \cdots \oplus_p X_n := \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \,\big|\, x_i \in X_i,\ 1 \leq i \leq n\},$$

equipped with the obvious inherited algebraic operations. The norm of an element in one of these spaces we denote $||\cdot||_{\oplus_p}$; we will only need the structures corresponding to $p = 1, \infty$, whose norms we define via:

$$||\mathbf{x}||_{\oplus_1} := \sum_{i=1}^{n} ||x_i||_{X_i}$$
$$||\mathbf{x}||_{\oplus_\infty} := \max_{1\leq i\leq n} ||x_i||_{X_i}$$

where the X_i's are suppressed in the norm subscript since it will be understood in context which spaces are forming the direct product. If, having fixed spaces X_i, we are considering a product of their respective dual spaces, the norm subscript will adopt an asterisk.

As a small final note, if $z \in \mathbb{C}$, $\mathrm{Re}(z) \in \mathbb{R}$ is its real part.

Lemma 4.1. *Let $X_1, X_2, \ldots, X_n$ be normed spaces. Then we have the following identification:*

$$(X_1 \oplus_1 X_2 \oplus_1 \cdots \oplus_1 X_n)^* \equiv X_1^* \oplus_\infty X_2^* \oplus_\infty \cdots \oplus_\infty X_n^*.$$

Specifically, each $\Psi \in (X_1 \oplus_1 X_2 \oplus_1 \cdots \oplus_1 X_n)^*$ is identified with an n-tuple

$$(\psi_1, \psi_2, \ldots, \psi_n) \in X_1^* \oplus_\infty X_2^* \oplus_\infty \cdots \oplus_\infty X_n^*$$

so that for each $\mathbf{x} \in (X_1 \oplus_1 X_2 \oplus_1 \cdots \oplus_1 X_n)$, it holds $\langle \Psi, \mathbf{x}\rangle = \sum_{i=1}^n \langle \psi_i, x_i\rangle$.

Proof. Define

$$S_1 : (X_1 \oplus_1 X_2 \oplus_1 \cdots \oplus_1 X_n)^* \to X_1^* \oplus_\infty X_2^* \oplus_\infty \cdots \oplus_\infty X_n^* : \Psi \mapsto (\psi_1, \psi_2, \ldots, \psi_n)$$

by the formula

$$\langle \psi_i, x\rangle := \left\langle \Psi, \underbrace{(0, 0, \ldots, x, \ldots, 0)}_{i\text{-th component}} \right\rangle,$$

for each $x \in X_i$, $1 \leq i \leq n$. Notice that for each $1 \leq i \leq n$, $x \in X_i$, it holds

$$|\langle \psi_i, x\rangle| \leq ||\Psi||_{\oplus_1^*} ||x||_{X_i}$$

Taking a max over the ψ_i's, we get $||S\Psi||_{\oplus_\infty} \leq ||\Psi||_{\oplus_1^*}$. Now define

$$S_2 : X_1^* \oplus_\infty X_2^* \oplus_\infty \cdots \oplus_\infty X_n^* \to (X_1 \oplus_1 X_2 \oplus_1 \cdots \oplus_1 X_n)^*$$

by the following formula:

$$\langle S_2\boldsymbol{\psi}, \mathbf{x}\rangle := \sum_{i=1}^{n} \langle \psi_i, x_i\rangle$$

for each $\mathbf{x} \in X_1 \oplus_1 X_2 \oplus_1 \cdots \oplus_1 X_n$, where $\boldsymbol{\psi} = (\psi_1, \psi_2, \ldots, \psi_n) \in X_1^* \oplus_\infty X_2^* \oplus_\infty \cdots \oplus_\infty X_n^*$. We then have the straightforward estimate $|\sum_{i=1}^n \langle \psi_i, x_i\rangle| \leq ||\boldsymbol{\psi}||_{\oplus_\infty^*} ||\mathbf{x}||_{\oplus_1}$. The verification that $S_2S_1 = \mathrm{Id}$ and $S_1S_2 = \mathrm{Id}$ is straightforward. □

Lemma 4.2. *Suppose X is a normed space over field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $W \subset X^\#$ is a subspace in the algebraic dual to X, $K \subset X$ is a nonempty convex, W-weakly closed subset, and $x_0 \in X$ is a point in X with $x_0 \notin K$. Then there exists $\phi \in W$ such that*

$$\mathrm{Re}(\phi(x_0)) < \inf_{y\in K} \mathrm{Re}(\phi(y))$$

Proof. This is a simple corollary of the general Hyperplane separation theorem for locally convex topological vector spaces over $\mathbb{F}$, see e.g. [3, p. 418, Cor. 11], first noting that X is a locally convex topological vector space when endowed with the W-weak topology whose topological dual is itself W [10, Thm. 2.4.11]. The Hyperplane separation theorem then yields a functional which separates the compact set $\{x_0\}$ and the convex, W-weakly closed set K as desired. □

Lemma 4.3. *Let X, Y be Banach spaces. If $T : X \to Y$ is an isometric isomorphism, then $T^* : Y^* \to X^*$ is also an isometric isomorphism.*

Proof. Since T is an isometric isomorphism, the adjoint mappings $T^*: Y^* \to X^*$ and $(T^{-1})^*: X^* \to Y^*$ are also bounded linear operators between the dual spaces of X, Y. All that remains to be checked is that $(T^{-1})^* = (T^*)^{-1}$, i.e. $(T^{-1})^* T^* = \mathrm{Id}_{Y^*}$ and $T^*(T^{-1})^* = \mathrm{Id}_{X^*}$. Regarding the former, we compute for $y^* \in Y^*$ and $y \in Y$,

$$\left\langle (T^{-1})^* T^* y^*, y \right\rangle = \left\langle T^* y^*, T^{-1} y \right\rangle = \left\langle Y^*, T T^{-1} y \right\rangle = \langle y^*, y \rangle.$$

whence the former quality holds. The reader is invited to check the latter. □

Lemma 4.4. *Let X be a Banach space, and suppose $A \subset X^*$ is weak-$*$ closed subspace. Then there exists a Banach space Z for which $A \equiv Z^*$. More precisely, $A \equiv (X/{}^\perp A)^*$ and the isometric isomorphism giving this identification is the adjoint π^* where $\pi : X \mapsto X/{}^\perp A$ is the canonical quotient mapping.*

For a proof of this fact, see [10, Thm 1.10.17], using the closed subspace ${}^\perp A$ and observing that the very identification constructed in the proof is the adjoint of the canonical quotient mapping as claimed.

Lemma 4.5. *Let X, Y be Banach spaces, and suppose $T : X \to Y$ is a bounded linear operator, with $T^* : Y^* \to X^*$. If $\mathcal{R}(T^*)$ is (norm) closed in X^*, then $\mathcal{R}(T^*)$ is weak-$*$ closed in X^*.*

Proof. By the First Isomorphism Theorem for Banach Spaces [10, 1.7.14], we can find an isomorphism $S : X/\ker(T) \to \mathcal{R}(T) \subset Y$ so that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi"'] & \mathcal{R}(T) \\ X/\ker(T) \arrow[ur, "S"'] & \end{tikzcd}$$

Taking adjoints, we obtain the following diagram:

$$\begin{tikzcd} X^* & \mathcal{R}(T)^* \arrow[l, "T^*"'] \arrow[dl, "S^*"] \\ (X/\ker(T))^* \arrow[u, "\pi^*"] & \end{tikzcd}$$

By Lemma 4.3, S^* is also an isomorphism; in particular, since S^* is surjective, $\mathcal{R}(T^*) = \mathcal{R}(\pi^*)$. By [10, 1.10.17],

$$\pi^* : (X/\ker(T))^* \to (\ker(T))^\perp \subset X^*$$

will be an isometric isomorphism, and by applying the preceding lemma and [10, Prop. 2.6.6(c)], we have

$$\mathcal{R}(T^*) = \mathcal{R}(\pi^*) = (\ker(T))^\perp = \overline{\mathcal{R}(T^*)}^{w^*}$$

so that $\mathcal{R}(T^*)$ is weak-$*$ closed. □

Theorem 4.6. *If $A_1, A_2, \ldots, A_n \subset X^*$ is a finite collection of weak-$*$ closed subspaces, and the sum $\sum_{j=1}^n A_j \subset X^*$ is (norm) closed, then $\sum_{j=1}^n A_j$ is weak-$*$ closed.*

Proof. By Lemma 4.4, for $1 \le j \le n$ we can find isometric isomorphisms $\pi_j^* : Z_j^* \to A_j$ where $Z_j = (X/{}^\perp A_j)$. Let

$$Z_1^* \oplus_\infty Z_2^* \oplus_\infty \cdots \oplus_\infty Z_n^*$$

be the direct sum of spaces Z_j^*, $1 \le j \le n$ equipped with the ℓ_∞ norm. Define the mapping

$$S : Z_1^* \oplus_\infty Z_2^* \oplus_\infty \cdots \oplus_\infty Z_n^* \to X^* : (z_1^*, z_2^*, \ldots, z_n^*) \mapsto \sum_{j=1}^n \pi_j^*(z_j^*).$$

Notice that by hypothesis, $\mathcal{R}(S) = \sum_{j=1}^{n} A_j \subset X^*$ is (norm) closed. To complete the proof by invoking Lemma 4.5, we need to produce an operator T for which $S = T^*$. Define

$$T : X \to Z_1 \oplus_1 Z_2 \oplus_1 \cdots \oplus_1 Z_n : x \mapsto (\pi_1(x), \pi_2(x), \ldots, \pi_n(x)).$$

where $(Z_1 \oplus_1 Z_2 \oplus_1 \cdots \oplus_1 Z_n)^* \equiv Z_1^* \oplus_\infty Z_2^* \oplus_\infty \cdots \oplus_\infty Z_n^*$. Given any $\psi \in (Z_1 \oplus_1 Z_2 \oplus_1 \cdots \oplus_1 Z_n)^*$, $x \in X$, we check

$$\langle T^*\psi, x\rangle = \langle \psi, Tx\rangle = \langle \psi, (\pi_1(x), \pi_2(x), \ldots, \pi_n(x))\rangle$$

For $1 \leq j \leq n$, we can find $\psi_j \in (X/^{\perp}A_j)^*$ so that ψ is identified with the n-tuple $(\psi_1, \psi_2, \ldots, \psi_n) \in Z_1^* \oplus_\infty Z_2^* \oplus_\infty \cdots \oplus_\infty Z_n^*$. That is,

$$\langle \psi, (\pi_1(x), \pi_2(x), \ldots, \pi_n(x))\rangle = \langle (\psi_1, \psi_2, \ldots, \psi_n), (\pi_1(x), \pi_2(x), \ldots, \pi_n(x))\rangle = \sum_{j=1}^{n} \psi_j \pi_j(x).$$

So, up to isometric isomorphism, when T^* is viewed as a mapping from $Z_1^* \oplus_\infty Z_2^* \oplus_\infty \cdots \oplus_\infty Z_n^* \to X^*$, we have that $T^* = S$. Since T^* is a bounded linear operator with closed range, by Lemma 4.5 its range, $\sum_{j=1}^{n} A_j$, is weak-$*$ closed. $\square$

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