A SURVEY OF RANDOM WALKS ON NETWORKS

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"PageRank can be thought of as a model of user behavior. We assume there is a 'random surfer' who is given a Web page at random and keeps clicking on links, never hitting 'back' but eventually gets bored and starts on another random page. The probability that the random surfer visits a page is its PageRank." Sergey Brin and Lawrenge Page on their search engine prototype Google[6].

1. Introduction

There is a famous saying in the mathematics community which captures quite nicely the spirit and beauty of the random walk process: "A drunk mathematician always stumbles home, but a butterfly flies forever." This sentence references to the recurrence and transience problem of the random walk in space, which dates back nearly a century to Polya[23] in 1921. The problem is posed as follows. A hypothetical "walker" begins at a fixed point in space, and as each minute passes, the walker moves one step in a particular direction (north, south, east, west) chosen at random with equal probability. In two dimensions, it can be shown that the walker is expected (with probability 1) to return to its initial position infinitely many times– hence, the drunk mathematician finding her way home. In three dimensions, wherein the walker is allowed to drift up and down in addition to the planar directions, it can be shown that the walker is expected to return to its initial position only finitely many times, drifting forever thereafter. We will explore this problem in detail in Section 3.

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In the years since, random walks have emerged as a powerful theoretical tool for modeling processes with behave and evolve probabilistically in a static space. Random walks have appeared everywhere from quantum mechanics and mathematical physics[20], to finance[18], signal processing[24], and beyond. One of the most famous applications of the model is to the Google PageRank algorithm[6]; here, the "static space" in which the walker is moving is the internet. Loosely speaking, the more often the walker is expected to visit a website, the higher "ranked" it becomes in the Google indexing hierarchy.

The purpose of this paper is to explore some selected random walk models posed on various types of networks. Mathematically, networks are collections of vertices (or nodes), connected by edges. In section 2, we set up the preliminary notions needed for the paper. In sections 3 and 4, we explore the very classical problems for infinite lattice networks and simple networks, respectively. Finally, in sections 5 and 6, we look at much more modern theory wherein the probabilistic framework of the walker and network (resp.) are reimagined.

2. Preliminaries

2.1. **Stochastic and Markov Processes.** Our probability notations will follow convention: $\mathbb{P}[\cdot], \mathbb{P}[\cdot|\cdot], \mathbb{E}[\cdot]$ denote probability measure, conditional probability, and expectation operator, respectively. The capital letter X will be used to denote random variables in general, $\{X_n\}_{n\geq 0}$ will denote sequences of random variables with the index $n \geq 0$ called the time step (mostly the braces will be omitted for convenience).

A *discrete-time stochastic process* is a sequence $\{X_n\}_{n\geq 0}$ of random variables taking values in a set $\mathcal{S}$, called the *state space* (for example, $\mathbb{N}, \mathbb{Z}$ or the vertices of a graph). We can think of X_n as 'jumping' or 'walking' between the elements of the state space subject to certain probabilistic conditions. If $\mathcal{S}$ is discrete, as it will be for our purposes, then X_n is called a *discrete-time discrete-space stochastic process*. We will use letters i, j, k to denote general elements of $\mathcal{S}$ as needed. A *Markov Process* is a stochastic process X_n for which

$$\mathbb{P}\left[X_{n+1} \in A \middle| X_k = x_k,\ k \leq n\right] = \mathbb{P}\left[X_{n+1} \in A \middle| X_n = x_n\right]$$

for all events A and $n \geq 0$. This formalizes the notion that a Markov process is only determined by the most recent position of the walker, and that past history does not contribute anything when coupled with more recent information.

A Markov process X_n is analyzed through its *one-step transition probabilities*:

$$p_{ij} := \mathbb{P}\left[X_n = j \middle| X_{n-1} = i\right]$$

which is independent of n since X_n is Markov. Similarly, we have the *m-step transition probabilities*:

$$p_{ij}^{(m)} := \mathbb{P}\left[X_n = j \middle| X_{n-m} = i\right].$$

To concretely relate the two we use the *one-step transition matrix* $\mathbf{P} := [p_{ij}]$. The probabilities are related through the transition matrix by the *Chapman-Kolmogorov equations*,

$$p_{ij}^{(m+\ell)} = \sum_{k\in\mathcal{S}} p_{ik}^{(m)} p_{kj}^{(\ell)} \tag{1}$$

for any $i, j \in \mathcal{S}$ and $m, \ell \geq 0$. In particular, this means that we can find the m-step probability transition matrix $\mathbf{P}^{(m)}$ by taking powers of the 1-step transition matrix; that is, $\mathbf{P}^{(m)} = \mathbf{P}^m$. For a proof, see e.g., [16, Prop. 3.2.1]. Transition matrices encode not just local information about the probabilities of transitioning between states, but also global information about how distributions evolve in time. Suppose $\boldsymbol{\mu} : \mathcal{S} \to \mathbb{R}_{\geq 0} : i \mapsto \mu_i$ is a mass distribution for the initial step of a Markov process X_n; i.e., $\sum_{i\in\mathcal{S}} \mu_i = 1$ and $\mathbb{P}[X_0 = i] = \mu_i$. We suggestively use subscripts here because if we treat $\boldsymbol{\mu} = [\mu_i]$ as a *row* vector, then

$$\mathbb{P}\left[X_1 = i \middle| \mathbb{P}[X_0 = i] = \mu_i\right] = (\boldsymbol{\mu}\mathbf{P})_i$$

for any $i \in \mathcal{S}$. A *stationary distribution* for the Markov process X_n, conventionally denoted $\boldsymbol{\pi}$, is a distribution for which

$$\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}.$$

A state j is said to be *accessible* from i if there exists $n > 0$ for which $p_{ij}^{(n)} > 0$, a property denoted by $i \to j$. If $i \to j$ and $j \to i$, the states are said to *communicate* and we write $i \leftrightarrow j$. A Markov process is called *irreducible* if all states $i, j \in \mathcal{S}$ communicate.

Let N_i represent the number of times a Markov process returns to a state i given that $X_0 = i$. The state i is called *recurrent* if $\mathbb{E}[N_i] = \infty$. It can be shown [16, Prop. 3.2.4] that a state i is recurrent if and only if $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$. A Markov process X_n is itself called recurrent if every state is recurrent.

Lemma 1. *All states of an irreducible discrete-time Markov process $\{X_n\}_{n\geq 0}$ on a finite state space $\mathcal{S}$ are recurrent.*

Proof. The proof is in the manner of [16, Prop. 3.2.5] with this particular formulation expressed as an exercise. First, note that X_n must contain at least one recurrent state: otherwise, every state would be visited only finitely many times, which cannot occur since X_n is on a finite state space over infinite time. We need to prove that recurrence is a *class property*, in the sense that if $i \in \mathcal{S}$ is recurrent and $i \leftrightarrow j$ for some $j \in \mathcal{S}$, then j is recurrent as well. Having proved this, the claim follows from the irreducibility of the process X_n. Assume in that manner that $i \leftrightarrow j$ for two states $i, j \in \mathcal{S}$ and that i is recurrent. Then there exist integers $k, \ell \geq 0$ for which $p_{ij}^{(k)} > 0$ and $p_{ji}^{(\ell)} > 0$. Notice that if $n \geq 0$, $p_{jj}^{(\ell+n+k)} \geq p_{ji}^{(\ell)} p_{ii}^{(n)} p_{ij}^{(k)}$, since X_n can return to j from j by, among other paths, getting to i, remaining there, and returning to j. Then it holds

$$\sum_{n=0}^{\infty} p_{jj}^{(n)} \geq \sum_{n=0}^{\infty} p_{jj}^{(\ell+n+k)} \geq p_{ji}^{(\ell)} p_{ij}^{(k)} \sum_{n=0}^{\infty} p_{ii}^{(n)} \to \infty$$

since i is recurrent itself. We conclude j is recurrent and the claim follows. □

We say that a state $i \in \mathcal{S}$ is *periodic with period d* if $p_{ii}^{(n)} = 0$ for each integer $n \geq 0$ which is not divisible by d, and where d is the largest such integer with this property. If $d = 1$ the state is *aperiodic*. A Markov process X_n is itself called (a)periodic if every state is (a)periodic.

Lemma 2. *If for some state $i \in S$ we have $p_{ii}^{(2)} > 0$ and $p_{ii}^{(3)} > 0$, then the state is aperiodic.*

Proof. Notice first that since any $n \geq 2$ may be expressed $n = 2k_1 + 3k_2$ for nonunique $k_1, k_2 \in \mathbb{N}$, it follows

$$p_{ii}^{(n)} = p_{ii}^{(2k_1+3k_2)} \geq p_{ii}^{(2k_1)} p_{ii}^{(3k_2)} \geq \left(p_{ii}^{(2)}\right)^{k_1} \left(p_{ii}^{(3)}\right)^{k_2} > 0$$

and the claim holds. □

It can also be shown [16, Def. 3.2.11] that periodicity is a class property; that is, if $i \leftrightarrow j$ and i is periodic, then so is j (with the same period). Note that periodicity and recurrence are not mutually exclusive. States which are both aperiodic and recurrent are called *ergodic*, with X_n ergodic if all of its states are ergodic. The last thing that we want to recall is the well-known Ergodic theorem, recalled and proved in [16, Thm. 3.2.1].

Theorem 3. *If X_n is an irreducible and ergoic Markov process, then it possesses a stationary distribution $\boldsymbol{\pi} = [\pi_j]$ satisfying*

(1) $\pi_j = \lim_{n\to\infty} p_{ij}^{(n)}$, *for each $j \in \mathcal{S}$, independent of $i \in S$.*
(2) $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$
(3) $\sum_{i\in\mathcal{S}} \pi_i = 1$

As a consequence of the Perron-Frobenius theorem, $\boldsymbol{\pi}$ is unique.

2.2. **Graph theory.** A *graph* or *network* is a pair of sets $G = (V, E)$ where V is a collection of *vertices*, *points*, or *nodes*; and E is a collection of *edges*. An *undirected* graph has edges of the form $\{i, j\}$, taken without direction; *directed* graphs have edges of the form (i, j). An *edge weight* is a map $\mu : E \to [0, \infty)$ associating a positive number to each edge. Combining a weight with a graph forms a *weighted graph*. We will mostly consider undirected, unweighted graphs here for simplicity and accessibility.

The number of vertices is denoted $|V|$, and the number of edges $|E|$. We generally use the letters i, j, k, v, w to denote individual nodes. A *simple graph* is undirected and satisfies $|V| < \infty$, no loops (edges of the form $\{i, i\}$), and no multiple edges (copies of the same edge in E). If two vertices $i, j \in V$ lie along an edge, they are called *adjacent* and we express it $i \sim j$; in the case of directed networks, this relation is not necessarily symmetric. The *in degree* of a vertex i, denoted $d_{\text{in}}(i)$ is the number of edges of the form $(\cdot, i)$. The *out degree* of a vertex, denoted $d_{\text{out}}(i)$ is the number of edges of the form $(i, \cdot)$. If G is undirected, these numbers are the same, and are called the degree of a vertex $d(i)$ or d_i.

The following lemma will be useful and is known as the *Handshaking lemma* or the *degree-sum formula*.

Lemma 4. *If G is simple, then $\sum_{i\in V} d(i) = 2|E|$.*

Proof. The proof is simple and relies on overcounting. Notice that summing over all of the degrees contributes twice the number of edges since you will count each edge once at both of its endpoints. □

A *path* of length n is an ordered list $(v_0, v_1, \dots, v_n)$ of $n + 1$ vertices such that $v_{j-1} \sim v_j$, $1 \le j \le n$. G is said to be *connected* provided that there exists a path connecting any two vertices in V. The graph G is said to be *bipartite* if the vertex set admits a decomposition into two nonempty disjoint sets $V = V_1 \cup V_2$, called the biparition, so that there are no edges contained strictly in either set; that is, every edge connects one vertex from V_1 to one from V_2. A useful characterization is the following lemma, whose proof we omit but can be found in [2, Th. 1.5.10].

Lemma 5. *A graph G is bipartite if and only if it contains no cycles of odd length.*

Here are some important matrices associated with graph structure which we will use. Define matrices $\mathbf{A}_G = [a_{ij}], \mathbf{D}_G = [d_{ij}]$ by:

$$a_{ij} := \begin{cases} 0 & i \nsim j \\ 1 & i \sim j \end{cases}, \quad d_{ij} = \begin{cases} 0 & i \neq j \\ d(i) & i = j \end{cases}, \quad 1 \le i \le |V|, 1 \le j \le |V|.$$

Here, the vertices are enumerated in an arbitrary but fixed way. Most important linear algebraic properties of these matrices are invariant under the action of permutations, hence independent of the particular enumeration.

The *combinatorial Laplacian* associated to G is the matrix Δ, with rows and columns indexed by some enumeration of V, defined by

$$\Delta_{ij} = \left\{ \begin{array}{cc} d(j) & i = j, \\ -1 & i \sim j, \\ 0 & \text{otherwise} \end{array} \right. .$$

If $f : V \to \mathbb{R}$ and is treated as a column vector, we may speak of its *combinatorial Laplacian* as the matrix product Δf. We then have the formula

$$(\Delta f)(u) = \sum_{v \sim u} \big(f(u) - f(v)\big). \tag{2}$$

The *simple random walk* G is the discrete-time stochastic process whose state space is V, subject to the following one-step transition probabilities:

$$p_{ij} = \begin{cases} 0 & i \nsim j, \\ \frac{1}{d_{\text{out}}(i)} & i \sim j. \end{cases}$$

That is, the walk jumps between adjacent vertices with equal probability of encountering any vertex adjacent to its current step. Notice that this process is Markov because the $(n+1)$-st step of the walker is determined *only* by its position at time n. When G is connected (and hence $\mathbf{D}$ invertible) A one-line computation verifies the identity

$$\mathbf{P} = \mathbf{D}^{-1}\mathbf{A}.$$

2.3. **Calculus and Linear Algebra.** We want to briefly set down some notes about notation for various calculus-related tools. If $f : [0, \infty) \to \mathbb{R}$ is a continuous function, we denote its Laplace transform $\widehat{f}(s) = \int_0^\infty f(t)e^{-st}dt$; inverse Laplace transform is denoted $\mathcal{L}^{-1}\{f\}(t)$. Convolution is denoted by $(f \star g)(t) = \int_0^t f(s)g(t-s)ds$ and is defined for sufficiently continous functions $f, g : [0, \infty) \to \mathbb{R}$. We also need the following useful theorem:

Theorem 6 (Final Value Theorem)**.** *If $f : [0, \infty) \to \mathbb{R}$ is bounded and integrable, then*

$$\lim_{x \to \infty} f(x) = \lim_{s \to 0^+} s\widehat{f}(s).$$

From [12, Th. 8.4.4] we have the workhorse Perron-Frobenius Theorem

Theorem 7 (Perron-Frobenius)**.** *Let A be an $n \times n$, $n \geq 2$ nonnegative and irreducible matrix. Let $\rho(A)$ be it spectral radius; that is, the maximum absolute value of its eigenvalues. Then*

(1) $\rho(A) > 0$,
(2) $\rho(A)$ is an algebraically simple eigenvalue of A,
(3) There is an unique real vector $x \in \mathbb{R}^n$ for which $Ax = \rho(A)s$ and $\sum_i x_i = 1$.

Ir/reducibility here is in the standard sense: a matrix is reducible if there is a permutation matrix relating A to new form with a 0 block in one of the off-diagonal corners of nonzero dimension[12, Def. 6.2.21]. Another characterization of this is the following: A is irreducible if $(\mathrm{Id} + A)^{n-1}$ is a matrix with strictly positive entries[12, Lm. 8.4.1].

3. Recurrence on Infinite Networks

A *locally finite infinite network* of the form $G = (V, E)$ is a simple network as before, but such that V is of countably infinite size, and with the additional property that each vertex $x \in V$ has finitely many neighbors. These networks come in many forms. Here, we will investigate two particular types: the d-dimensional lattice, and the k-th rooted infinite tree.

In the first subsection will prove Polya's classical recurrence result which identifies the dimensions in which a walker moving randomly on a lattice, having started at the origin, is expected to return. We will prove this result in the cases $d = 1, 2, 3$ using purely combinatorial methodology. In the second subsection, we will put forth a generalized framework for investigating the recurrence and transience of random walks on locally finite infinite networks which utilizes what we will call energy methods, adapted from the literature of electrical networks[9, 5].

3.1. Recurrence on Lattices with Combinatorial methods.

Random walks on lattices are an interesting special case of the general theory of random walks on networks for a multitude of reasons. First, lattices of the form $\mathbb{Z}^d$ where $d \geq 1$ can show up in finite element methods as convenient discrete spatial models. Second, they are one of the handful of well-structured infinite networks on which the recurrence or transience of the symmetric walk has a well-defined, classically proved answer. In this section, we will cover the classic historical result due to Polya[23] in 1921. The recurrence/transience properties are known for walks on lattices of all dimensions; here, we prove the result up to dimension three.

Theorem 8. *The simple symmetric random walk on the lattice $\mathbb{Z}^d$ is recurrent for $d = 1, 2$ and transient for $d > 3$.*

Precisely, by the lattice $\mathbb{Z}^d$ we mean the set of points

$$\mathbb{Z}^d := \{(n_1, n_2, \dots, n_d) : n_j \in \mathbb{Z} \text{ for } 1 \leq j \leq d\};$$

these points are connected by an edge if and only if they are exactly one unit apart. Generally we denote points in this space by $\mathbf{x}, \mathbf{y}$ with the origin $\mathbf{0} = (0, 0, \dots, 0)$. The simple random walk X_n on the lattice is then interpreted as the walk beginning at the origin and then moving in discrete time subject to the one-step transition probabilities

$$\mathbb{P}\left[X_{n+1} = \mathbf{y} | X_n = \mathbf{x}\right] = \begin{cases} \frac{1}{2d} & \text{if } \mathbf{x} \sim \mathbf{y} \\ 0 & \text{if } \mathbf{x} \nsim \mathbf{y} \end{cases}.$$

To prove this in dimensions $d = 1, 2$ we will make use of the well-known Stirling's formula, which is

$$n! \approx \sqrt{2\pi n} e^{-n} n^n, \;\; n \geq 1,$$

interpreted in the sense that $\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n} e^{-n} n^n} = 1$. Also useful will be Vandermonde's combinatorial identity, which states that for m, n, r nonnegative, we have

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}.$$

Proof. The argument here follows [15]. We begin with the case of dimension one. Recall that the walk will be recurrent if and only if the expected number of visits of X_n to the origin, given $X_0 = 0$, is infinity. If

$$u_n = \mathbb{P}\left[X_n = 0 | X_0 = 0\right],$$

then

$$\mathbb{E}\left[\#\text{ visits to the origin } |X_0 = 0\right] = \sum_{n=1}^{\infty} u_n.$$

The number of paths originating at 0 and returning in $2n$ steps $\binom{2n}{n,n}$; there are no such paths of odd length. Any one of these occurs with probability $\left(\frac{1}{2}\right)^{2n}$, so via Sterling's formula,

$$u_{2n} = \binom{2n}{n,n}\left(\frac{1}{2}\right)^{2n} = \frac{(2n)!}{2^{2n}n!n!} \approx \frac{\sqrt{4\pi n}e^{2n}(2n)^{2n}}{e^{2n}(2\pi n)n^{2n}2^{2n}} = \frac{1}{\sqrt{\pi n}}$$

so that $\sum_{n=1}^{\infty} u_n \to \infty$.

In dimension two, there are $\sum_{k=0}^{n}\binom{2n}{k,k,n-k,n-k}$ paths of length $2n$ originating and terminating at $\mathbf{0}$.

$$u_{2n} = \frac{1}{4^{2n}}\sum_{k=0}^{n}\frac{(2n)!}{k!k!(n-k)!(n-k)!} = \frac{1}{4^{2n}}\frac{(2n)!}{n!n!}\sum_{k=0}^{n}\binom{n}{k}^2.$$

By Vandermonde's identity,

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n},$$

so

$$u_{2n} = \frac{1}{4^{2n}}\binom{2n}{n} = \frac{1}{4^{2n}}\frac{(2n)!^2}{(n!)^4} \approx \frac{1}{4^{2n}}\frac{(4\pi n)(2n)^{4n}}{e^{4n}}\frac{e^{4n}}{(2\pi n)^2 n^{4n}} = \frac{1}{\pi n}$$

so that $\sum_{n=1}^{\infty} u_n \to \infty$.

Having established the recurrence of the random walks in dimensions one and two, we will now prove that the walk is transient in dimension three. In this case, the walker has $\binom{2n}{k,k,j,j,n-k-j,n-k-j}$ possible routes with k steps left/right, j steps up/down, and $n-k-j$ steps forward/backward, each occurring with a probability $\frac{1}{6^{2n}}$. This means

$$u_{2n} = \frac{1}{6^{2n}}\sum_{j+k\leq n}\frac{(2n)!}{j!^2k!^2(n-j-k)!^2} = \frac{1}{2^{2n}}\binom{2n}{n}\sum_{j+k\leq n}\left(\frac{(n)!}{3^{2n}j!k!(n-j-k)}\right)^2$$

The quantity $\left(\frac{(n)!}{j!k!(n-j-k)}\right)$ is maximized when n, j, k are as close together as possible, so

$$u_{2n} \leq \frac{1}{2^{2n}}\binom{2n}{n}\left(\frac{n!}{3^n\left(\frac{n}{3}\right)!}\right)\sum_{j+k\leq n}\left(\frac{(n)!}{3^n j!k!(n-j-k)}\right).$$

The rightmost piece of the inequality is a distribution; e.g., for the outcomes of drawing one ball at a time without replacement from an urn containing n balls, $j, k, n-j-k$ of which are the same color; i.e.,

$$u_{2n} \leq \frac{1}{2^{2n}}\binom{2n}{n}\left(\frac{n!}{3^n\left(\frac{n}{3}\right)!}\right).$$

Sterling's formula then yields $u_{2n} \approx \frac{K}{n^{3/2}}$ which forces $\sum_{n=1}^{\infty} u_n < \infty$. □

4. Convergence on Simple Networks

Here is the central question: what conditions on the geometric structure of a simple graph are necessary to ensure the ergodicity of the symmetric random walk, and subsequently, when a stationary distribution does exist, at what rate do the transitional probabilities converge to the stationary distribution? The answer to the former is classical and well-known. An elementary estimate of the rate of convergence to the stationary distribution in terms of the eigenvalues of a normalized Laplacian matrix for the graph, due to Lovasz[17], is given.

Theorem 9. *If G is simple, connected, and not bipartite, then the simple random walk is irreducible, recurrent, and aperiodic. The stationary distribution as in Theorem 3 is given by*

$$\pi_i = \frac{d_i}{2|E|}, 1 \leq i \leq |G|.$$

Proof. The irreducibility and recurrence of the simple random walk process will follow from the connectedness condition, and the aperiodicity will follow from the non-biparteness condition. Let $i, j \in V$ be fixed; since G is connected we can find a path of length $m \geq 1$ of the form

$$(i = v_0, v_1, \ldots, v_m = j)$$

connecting i to j. Notice that the this path will occur with positive probability; i.e.,

$$p_{ij}^{(m)} \geq \mathbb{P}\left[X_1 = v_1, \ldots, X_m = v_m | X_0 = v_0\right] = \prod_{\ell=1}^{m} \frac{1}{d(v_{\ell-1})} > 0.$$

That is, $i \to j$. A symmetrically identical argument will show that i is accessible from j by looking at the walk along this same path, but in reverse. We conclude that $i \leftrightarrow j$ for every two vertices i, j and that the simple random walk is irreducible, from which Lemma 1 then yields the recurrence of the process.

Now, let us assume that the random walk is periodic with period $d > 1$. By Lemma 2 and the irreducibility of the simple random walk, either $p_{ii}^{(2)} = 0$ or $p_{ii}^{(3)} = 0$ holds for every vertex $i \in V$. It is clear that the probability of returning to a particular vertex must be positive, since both jumping to neighbor and then jumping back occur with positive transitional probability. In fact, $p_{ii}^{(2k)} > 0$ for any $k \geq 1$ and $i \in V$. This means that $p_{ii}^{(3)} = 0$ and that $d = 2$, while possibly not the largest such integer, satisfies the definition for periodicity. In particular, $p_{ii}^{(2k+1)} = 0, k \geq 0$. Since the random walk jumps between neighbors with equal probability, the only way for $p_{ii}^{(2k+1)} = 0$ is for the graph to contain no cycles of odd length; via Lemma 5, this means G is bipartite.

Since the Perron-Frobenius theorem guarantees that the stationary distribution is unique, we simply need to check that the claimed formula satisfies each of the necessary conditions. First note

$$\begin{aligned}(\pi \mathbf{P})_j &= \sum_{i=1}^{n} \pi_i p_{ij} = \sum_{i=1}^{n} \frac{d_i}{2|E|} p_{ij} \\ &= \frac{1}{2|E|} \sum_{i \sim j} d_i \frac{1}{d_i} = \frac{d_j}{2|E|} = \pi_j\end{aligned}$$

where $n := |V|$. Also, checking the normalization condition,

$$\sum_{j=1}^{n} \pi_j = \sum_{j=1}^{n} \frac{d_j}{2|E|} = \frac{1}{2|E|} \sum_{j=1}^{n} d_j = \frac{2|E|}{2|E|} = 1.$$

□

Our second and final result concerns estimating the rate of convergence of the transitional probabilities $p_{ij}^{(n)}$ to the stationary distribution π_j. This proof follows that given by Lovasz[17] with some notational adjustments. Recall from the introductory section that we expressed

$$\mathbf{P} = \mathbf{D}^{-1}\mathbf{A}$$

Let $\mathbf{D}^{1/2}, \mathbf{D}^{-1/2}$ be the matrices defined, in the obvious manner, to contain the square-roots of the degrees of the vertices and their reciprocals, respectively. Again, this makes sense when G is connected and $|G| \geq 2$. Then we have

$$\mathbf{P} = \mathbf{D}^{-1/2}\left(\mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}\right)\mathbf{D}^{1/2}.$$

Define $\mathbf{N} = \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$. This matrix is very close in structure to the well-studied *normalized symmetric Laplacian matrix*; Chung's *Spectral Graph Theory* [8] is an classic, detailed study of matrices like these. $\mathbf{N} = [N_{ij}]$ is symmetric, so we can find $n := |V|$ possibly non-unique real eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, each with an associated normalized eigenvector $\mathbf{f}_k = [f_{k\ell}]_{\ell=1}^{n}$, $1 \leq k \leq n$ which together are pairwise orthogonal; in particular,

$$\mathbf{N} = \sum_{\ell=1} \lambda_\ell \mathbf{f}_\ell \mathbf{f}_\ell^T.$$

Theorem 10. *Let G be connected and non-bipartite, and $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$ as above. Then we have*

$$|p_{ij}^{(t)} - \pi_j| \leq \lambda^t \sqrt{\frac{d_j}{d_i}}.$$

for each $t \geq 1$.

Proof. First note with a one-line computation that the vector $\mathbf{g} = [g_i] = [\sqrt{d_i}]$ is a left eigenvector with eigenvalue 1. By the Perron-Frobenius theorem, since $\mathbf{g}$ is strictly positive, this is the dominant eigenvector whose eigenvalue only appears once and is largest in absolute value (in particular, $|\lambda_k| < 1$ for $2 \leq k \leq n$); i.e., $\lambda_1 = 1$ and $f_{1\ell} = \sqrt{\frac{d_\ell}{2|E|}}$. We compute for $t \geq 1$

$$\mathbf{P}^t = \mathbf{D}^{-1/2}\mathbf{N}^t\mathbf{D}^{1/2} = \sum_{\ell=1}^{n} \lambda_\ell^t \mathbf{D}^{-1/2}\mathbf{f}_\ell\mathbf{f}_\ell^T\mathbf{D}^{1/2}.$$

Entry-wise,

$$p_{ij}^{(t)} = \sum_{\ell=1}^{n} \lambda_\ell^t f_{\ell i} f_{\ell j} \sqrt{\frac{d_j}{d_i}} = f_{1i} f_{1j} \sqrt{\frac{d_j}{d_i}} + \sum_{\ell=2}^{n} \lambda_\ell^t f_{\ell i} f_{\ell j} \sqrt{\frac{d_j}{d_i}} = \frac{d_j}{2|E|} + \sum_{\ell=2}^{n} \lambda_\ell^t f_{\ell i} f_{\ell j} \sqrt{\frac{d_j}{d_i}}.$$

That is,

$$|p_{ij}^{(t)} - \pi_j| = \left| \sum_{\ell=2}^{n} \lambda_\ell^t f_{\ell i} f_{\ell j} \sqrt{\frac{d_j}{d_i}} \right| \leq \lambda^t \sqrt{\frac{d_j}{d_i}} ||\mathbf{F}_{(\cdot,i)}|| ||\mathbf{F}_{(\cdot,j)}|| = \lambda^t \sqrt{\frac{d_j}{d_i}}.$$

In particular, $p_{ij}^{(t)} \to \pi_j$ as claimed before. The matrix $\mathbf{F} = [f_{\ell i}]_{\ell,i}$, as a formality, is the square matrix whose rows consist of the orthonormal system $\mathbf{f}_\ell$ $1 \leq \ell \leq n$. Then the summand is recognized as the real inner product of the i-th and j-th columns. Since $\mathbf{F}$ is orthogonal, the columns also form an orthonormal system, which when equipped with Cachy-Schwarz inequality facilitates the last step. □

5. Harmonic Functions and the Simple Asynchronous Walk in Continuous Time

In the preceding section we examined the stationary distribution associated with a random walk on a simple network taking place in discrete time. One straightforward way to broaden the applicability of the model while keeping the analysis manageable is to allow the walker to take asynchronous steps in continuous time. Still jumping between adjacent nodes, the walker now hesitates at each step for some amount of time. The transitional probabilities of the walker remain unchanged; however, the amount of time that a walker spends at a particular node is no longer deterministic in discrete time steps. Rather, we introduce a Poisson-type *waiting time distribution* (WTD). Having reached node i at time t_0, the time of next transition is subject to the WTD mass function

$$\mathbb{P}\left[X_{t_0+t} = i \middle| X_{t_0} = i\right] = \lambda_i e^{-\lambda_i t}, \qquad t > 0,$$

where the *local intensity* $\lambda_i > 0$ can be chosen as desired (and not necessarily in a uniform fashion). As in [22], letting $p_i(t) = \mathbb{P}[X_t = i]$ and $\mathbf{p} = [p_1(t)\ p_2(t) \dots p_N(t)]^T$, where $N = |V|$, be the probabilities that the process is at state i at time $t > 0$, then we have the following rate equation:

$$\frac{dp_i}{dt} = \sum_{j \sim i} \frac{\lambda_j}{d_j} p_j(t) - \frac{\lambda_i}{d_i} p_i(t).$$

One can think of this as a scaled difference equation measuring the total in/out flow of the probability at a vertex. Interestingly, when the local intensities are chosen to be proportional to degrees, e.g., $\lambda_i = d_i$, the rate equations reduce to

$$\frac{d\mathbf{p}}{dt} = \Delta\left(\mathbf{p}(t)\right).$$

If we want to find a steady state solution to this process, we then need to solve the Dirichlet-type problem

$$\Delta\left(\mathbf{p}(t)\right) = 0,$$

which motivates the following theorems.

If $H \subset V$ is a subset of vertices, we can define its boundary

$$\partial H := \{x \in V : x \notin H,\ \ \exists y \in H \text{ s.t. } x \sim y\},$$

with the closure set $\overline{H} = H \cup \partial H$. This concept is illustrated in Figure 1.

Consider the following Dirichlet-type boundary value problem, where $H \subset V$ is a fixed subset of vertices serving as the region of interest, and $\phi : \partial H \to \mathbb{R}$ is given.

$$\begin{cases} (\Delta f)(u) = 0 & u \in H \\ f(u) = \phi(u) & u \in \partial H \end{cases} \tag{3}$$

The solution to this problem, as well as its more general Poisson-type cousin, are known and have been well studied [7]. Any such f is said to be *harmonic* on H.

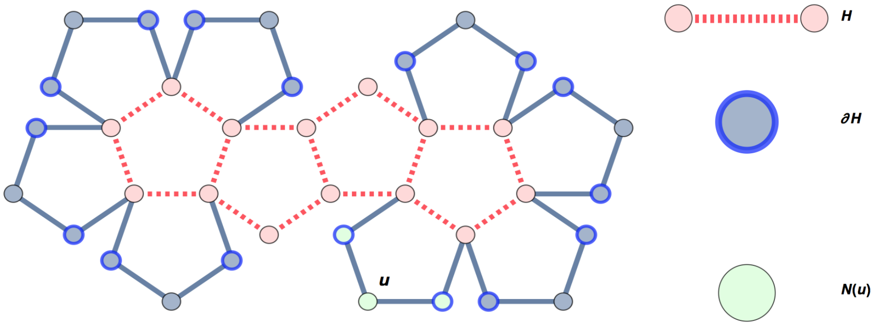

FIGURE 1. A dodecaheral graph with subset H, its vertex boundary ∂H, and a particular vertex neighborhood $N(u)$. (Borrowed from an old manuscript of mine.)

Theorem 11 (Maximum Principle). *Assume $G = (V, E)$ is connected and that $H \subset V$ is nonempty. Let f be a solution to the Problem (3). Then*

$$\max_{u \in \overline{H}} f(u) = \max_{u \in \partial H} f(u).$$

Proof. If f is constant the claim is clear. Assuming otherwise and arguing for contradiction, identify one such vertex $u^* \in H$ achieving the maximum of f on $\overline{H}$, which has a neighbor v^*, possibly in ∂H, such that $f(v^*) < f(u^*)$. Also, we know that $f(v) \leq f(u^*)$ for each $v \sim u^*$. Then

$$\begin{aligned}(\Delta f)(u^*) &= \sum_{v \sim u^*} f(u^*) - f(v) \\ &= f(u^*) - f(v^*) + \sum_{\substack{v \sim u \\ v \neq v^*}} f(u^*) - f(v) > 0,\end{aligned}$$

proving that f cannot simultaneously have a maximum on the interior and satisfy problem (3). □

A symmetrically identical argument will suffice to show

$$\min_{u \in \overline{H}} f(u) = \min_{u \in \partial H} f(u).$$

Theorem 12. *If $G = (V, E)$ is connected and simple, the only solutions $f : V \to \mathbb{R}$ to the equation*

$$\Delta f \equiv 0 \tag{4}$$

are constant functions.

Proof. Clearly constant functions solve this equation. Suppose for a moment that f solves equation (4) and $f(u_0) = a$ for some vertex u_0 and $a \in \mathbb{R}$. Let $g : V \to \mathbb{R}$ be the constant function $g(u) = a$ for each $u \in V$. Since both f, g are harmonic, so is their difference. In particular, $f - g$ solves the equation (3) with choice $H = V \backslash \{u_0\}$, $\phi(u_0) = 0$. Since G is connected, $\partial H = \{u_0\}$, so via the preceding remarks, $f - g$ attains both its maximum and minimum at this vertex where their difference is 0, i.e., $f \equiv g$ forcing f to be constant as well. □

If G is a disconnected network, one can apply the preceding result on each of its connected components to obtain solutions that are instead component-wise constant.

Returning to the asynchronous walk, our search for a steady state distribution led to the global Dirichlet problem

$$\Delta(\mathbf{p}(t)) \equiv 0.$$

We now know that this can only occur if $\mathbf{p}(\mathbf{t}) \equiv c\mathbf{1}$ for some $c \in \mathbb{R}$, where $\mathbf{1}$ is the column vector with every entry set to 1. Employing a normalization condition, this forces $p_i(t) = \frac{1}{N}$ for all $t > 0$. This is interesting because the solution is independent of the geometry of the network; yielding to a uniform steady-stade distribution in all cases.

6. Temporal Networks and the Active Random Walk

In this section, we will reimagine the spatial structure on which the walk is taking place. The framework we will use is known as a *temporal network*. The main idea behind temporal networks is that static networks; i.e., those whose structures do not evolve in time, are insufficient to model many dynamical systems whose dynamics change in time. To illuminate this, consider for example a population network of people amongst whom an infectious disease is spreading, edges between people can indicate transmission of disease, which is by no means a static phenomenon[10]. Edges in the network are no longer deterministic components of the structure, but rather subject to stochastic behavior, turning on and off subject to the forces within the contextual framework. There are of course many angles from which to approach the mathematical formalism, but we will explore the remarkable avenue developed in 2012 by authors Hoffman, Porter, and Lambiotte[10] which is known as the *active random walk*. At the center is a generalized master equation for the process based on the Montroll-Weiss (MW) equation from statistical mechanics[20]. Like many results in this field, MW was developed for and by physicists in the context of lattices, and then has re-emerged in recent years as network scientists across a number of disciplines have sought and proved generalizations of the physical theories for lattices to other networks of various flavors and designs.

The first order of business will be to explain the components of the model, after which we will derive the generalized MW equation and then the steady-state solution to the process in the special case where the WTDs are Poisson. This will all follow [10].

As we encountered earlier, to each network we may associate an adjacency matrix $\mathbf{A}$ whose entries indicate the static presence of edges. Here, within a fixed ambient network $G = (V, E)$ with $|V| =: n$ (we are choosing G unweighted and undirected for simplicity), we define an $n \times n$ matrix $\mathbf{\Psi} = [\psi_{ij}(t)]$ whose entries $\psi_{ij}(t)$, each a WTD, give the probability of an edge appearing between nodes i and j between time t and $t + dt$. Globally this means that at any given time, the probability of any edges being present in the network is 0. We can think of them as 'lighting up' at random moments in time. The walker, subjected to some initial condition, then moves between incident vertices instantaneously as soon as a neighbor becomes available. There is also a renewal component to this model; as soon as the walker transitions to a new vertex, all of the incident edges have their WTD's reinitialized. Since each ψ is a distribution, we have

$$\int_0^\infty \psi_{ij}(t)dt = 1.$$

Similarly the probability that an edge does appear between i and j in time $[0, t]$ is

$$\int_0^t \psi_{ij}(s)ds,$$

and the probability that an edges does not appear in this time window is

$$\chi_{ij}(t) := 1 - \int_0^t \psi_{ij}(s)ds. \tag{5}$$

Assuming for a moment that the walker has arrived at node j, we want to calculate the probability T_{ij} that it transitions to a neighboring node i. If i lies across a single edge between the two nodes, this is of course $T_{ij} = \psi_{ij}$. An important distinction occurs when i is one of many nodes leaving j; in this case, T_{ij} is a weighted probability; both an edge need appear between them at time t and all other edges should stay turned off during $[0, t)$; i.e.,

$$T_{ij}(t) = \psi_{ij}(t) \prod_{\substack{k\sim j \\ k\neq i}} \chi_{kj}(t) = -\frac{d\chi_{ij}(t)}{dt} \prod_{\substack{k\sim j \\ k\neq i}} \chi_{kj}(t). \tag{6}$$

From the functions T_{ij} we can construct the *effective transition matrix*, $\mathbb{T} = [\rho_{ij}]$, whose entries

$$\rho_{ij} = \int_0^\infty T_{ij}(t)dt$$

yield a total probability of transitioning from j to i. Also useful is the *continuous transition matrix* $\mathbf{T} = [T_{ij}]$. The probability that the walker is at node i at a given instance in time t is denoted $p_i(t)$; it is the integral over the probabilities $q_i(s)$ of having arrived at time $s \leq t$ weighted by the probability $\phi_i(t-s)$ of not having left since then, i.e.,

$$p_i(t) = \int_0^t \phi_i(t-s)q_i(s)ds. \tag{7}$$

In this light, we can express the probabilities p in Laplace space as the product

$$\widehat{p_i}(s) = \widehat{\phi_i}(s)\widehat{q_i}(s). \tag{8}$$

This is the starting point for the generalized MW equation. If we let

$$T_i(t) = \sum_{j\sim i} T_{ji}(t) \tag{9}$$

be the probability that the walker leaves i between times t, $t + dt$, then

$$\phi_i(t) = 1 - \int_0^t T_i(s)ds,$$

or in Laplace space, $\widehat{\phi_i}(s) = \frac{1}{s}(1 - \widehat{T_i}(s))$. The other component $q_i(t)$ in equation (7), can be found by summing over the probabilities $q_i^{(k)}(t)$ of reaching node i at time t through exactly k steps:

$$q_i(t) = \sum_{k=0}^{\infty} q_i^{(k)}(t).$$

These are related by the recursion relation

$$q_i^{(k+1)}(t) = \int_0^t \left[\sum_{j\sim i} T_{ij}(t-\tau) q_j^{(k)}(\tau) \right] d\tau,$$

which computes a probability of arriving in $k+1$ steps as the product of probabilities of arriving to neighbors of i in k steps weighted by the probability of transitioning along each edge. In Laplace space,

$$\widehat{q}_i^{(k+1)}(s) = \sum_{j\sim i} \widehat{T}_{ij}(s) q_j^{(k)}(s).$$

Summing over k and adding $\widehat{q}_i^{(0)}(s)$,

$$\widehat{q}_i^{(0)}(s) + \sum_{k=0}^{\infty} \widehat{q}_i^{(k+1)}(s) = \sum_{j\sim i} \widehat{T}_{ij}(s) \sum_{k=0}^{\infty} q_j^{(k)}(s) + \widehat{q}_i^{(0)}(s).$$

In matrix form,

$$\widehat{\mathbf{q}}(s) = \widehat{\mathbf{T}}\widehat{\mathbf{q}}(s) + \widehat{\mathbf{q}}^{(0)}(s),$$

where $\mathbf{q} = [q_1(t)\ q_2(t) \dots q_n(t)]^T$. Since $\mathbf{q}^{(0)}(t) = \mathbf{p}(0)\delta(t)$, it holds $\widehat{\mathbf{q}}^{(0)}(s) = \mathbf{p}(0)$, yielding

$$\widehat{\mathbf{q}}(s) = (\mathrm{Id} - \mathbf{T})^{-1}\, \widehat{\mathbf{q}}^{(0)}(s).$$

Returning to equation (8), we obtain the generalized MW equation

$$\widehat{p}_i(s) = \frac{1}{s}(1 - \widehat{T}_i(s)) \sum_{k=1}^{n} \left(\mathrm{Id} - \widehat{T}(s)\right)^{-1}_{ik} p_k(0),$$

which takes the Matrix form

$$\widehat{\mathbf{p}}(s) = \frac{1}{s}\left(\mathrm{Id} - \widehat{\mathbf{D}}(s)\right)\left(\mathrm{Id} - \widehat{\mathbf{T}}(s)\right)^{-1} \mathbf{p}(0) \tag{10}$$

where $(\widehat{\mathbf{D}})_{ij}(s) = \widehat{\mathbf{T}}_i(s)\delta_{ij}$. Omitting a full derivation[10], by differentiating the MW equation and using convolution, one can obtain the integro-differential master equation for the active random walk:

$$\frac{d\mathbf{p}}{dt} = \left(\mathbf{T}(t) \star \mathcal{L}^{-1}\{\widehat{\mathbf{D}}^{-1}(s)\} - \delta(t)\right) \star \mathbf{K}(t) \star \mathbf{p}(t). \tag{11}$$

The star symbol is emboldened to emphasize that the matrix-vector products in the equation are taken in the usual manner with the function convolution operation rather than the scalar multiplication operation. Also, here, $\mathbf{K}(t)$ is called a *memory kernel*, defined in Laplace space by

$$\widehat{\mathbf{K}}(s) = \left(s\widehat{\mathbf{D}}(s)\right)\left(\mathrm{Id} - \widehat{\mathbf{D}}(s)\right)^{-1}.$$

This kernel characterizes the amount of memory in the system at a moment in time[3], and is usually dependent on probabilities spanning over a nonvanishing amount of time. This means that the dynamics of the system are generally non-Poissonian in nature; the exception to the rule is when the WTD are themselves Poisson, in which case $\mathbf{K}(t) = \delta(t)\mathrm{Id}$.

Looking towards the steady-state solution, one worthwhile point to reexamine is the distribution $T_i(t)$, the probability that the walker leaves node i between times t, $t+dt$. As long as the graph is connected and the WTDs between adjacent edges are nonzero, we expect that $\int_0^\infty T_i(t)dt = 1$ for any $i \in V$, confirming that the walker is expected to transition from any vertex in infinite time. So using equations (6) and (9) we have

$$T_i(t) = \sum_{j\sim i} T_{ji}(t) = -\sum_{j\sim i} \frac{d\chi_{ji}(t)}{dt} \prod_{\substack{k\sim j \\ k\neq i}} \chi_{jk}(t) = -\frac{d}{dt}\left(\prod_{j\sim i} \chi_{ji}(t)\right) \tag{12}$$

forcing

$$\int_0^\infty T_i(t)dt = -\left[\prod_{j\sim i}\chi_{ji}(t)\right]_{t=0}^{\infty} = 1. \tag{13}$$

It is truly lucky that despite working within a dynamical system governed by an intimidating integro-differential master equation, in the general case, it is possible to obtain an analytical expression for the steady-state solution to equation (11). The derivation[10] is accessible, but we will not reproduce it in full here. At the heart of the approach to the solution is an application of the final value theorem to the MW equation (10). As in the case of the simple random walk on networks, we expect there to exist a stationary distribution $\mathbf{p}^*$ as a limit of $\mathbf{p}(t)$ as long as the graph is connected. By the final value theorem and equation (10), one has

$$\begin{aligned}\mathbf{p}^* &= \lim_{t\to\infty}\mathbf{p}(t) = \lim_{s\to 0^+} s\widehat{p}(s)\\ &= \lim_{s\to 0^+}\left(\mathrm{Id}-\widehat{\mathbf{D}}(s)\right)\left(\mathrm{Id}-\widehat{\mathbf{T}}(s)\right)^{-1}\mathbf{p}(0)\end{aligned}$$

The search for $\mathbf{p}^*$ then becomes a search for the dominant eigenvector of $(\mathrm{Id}-\widehat{\mathbf{D}}(s))(\mathrm{Id}-\widehat{\mathbf{T}}(s))^{-1}$, which as an operator maps the initial distribution to the steady-state distribution. As it turns out, this eigenvector coincides with the least dominant eigenvector, with eigenvalue 0, of the matrix $\mathbb{C} := (\mathrm{Id}-\mathbb{T})\,\overline{\mathbf{D}}^{-1}$, where $\overline{\mathbf{D}}_{ij} := \mathbb{E}\left[T_j\right]\delta_{ij}$, which is invertible in practice since the graph is assumed connected. Notice that as we confirmed in equation (13), the effective transition matrix $\mathbb{T}$ is in fact stochastic; via Perron-Frobenius, this guarantees the existence of a unique eigenvector $\mathbf{x}$ with eigenvalue 1. In turn, the steady state solution is

$$\mathbf{p}^* = \overline{\mathbf{D}}\mathbf{x}, \tag{14}$$

since $\mathbb{C}\mathbf{p}^* = (\mathrm{Id}-\mathbb{T})\,\overline{\mathbf{D}}^{-1}\overline{\mathbf{D}}\mathbf{x} = \mathrm{Id}\mathbf{x} - \mathbb{T}\mathbf{x} = 0$.

6.1. **Example: The Poisson Case.** Continuing in the manner of Hoffman, Porter, Lambiotte[10], an interesting case study of the general theory presented above occurs when we choose the WTDs of the edges to be Poisson; that is,

$$\psi_{ij}(t) = \lambda_{ij}e^{-\lambda_{ij}t}, \qquad \lambda_{ij}, t \geq 0.$$

Combining equations (5) and (6), we get

$$T_{ij}(t) = \lambda_{ij}e^{-\lambda_{ij}t}\prod_{\substack{k\sim j\\ k\neq i}}\left(1-\int_0^t \lambda_{kj}e^{-\lambda_{kj}t}\right) = \lambda_{ij}e^{-\Lambda_j t}, \tag{15}$$

where $\Lambda_j = \sum_{i\sim j}\lambda_{ij}$. Similarly $T_j(t) = \Lambda_j e^{-\Lambda_j t}$, which gives $\widehat{T}_j(s) = \frac{\Lambda_j}{\Lambda_j+s}$ and as a result $[\widehat{\mathbf{D}}(s)^{-1}]_{ij} = \left(\frac{\Lambda_j}{s+\Lambda_j}\right)\delta_{ij}$ and $[\widehat{\mathbf{D}}(s)^{-1}]_{ij} = \left(1+\frac{s}{\Lambda_j}\right)\delta_{ij}$. We can then evaluate the memory kernel in Laplace space:

$$[\widehat{\mathbf{K}}(s)]_{ij} = \frac{s\frac{\Lambda_j}{s+\Lambda_j}}{1-\frac{\Lambda_j}{s+\Lambda_j}}\delta_{ij} = \Lambda_j\delta_{ij} \tag{16}$$

Evaluating $\mathcal{L}^{-1}\{\widehat{\mathbf{D}}^{-1}(s)\}_{ij} = \left(\delta(t)+\frac{\delta'(t)}{\Lambda_j}\right)\delta_{ij}$. Similarly, $[\mathbf{K}(t)]_{ij} = \mathcal{L}^{-1}\{\Lambda_j\delta_{ij}\} = \Lambda_j\delta(t)\delta_{ij}$. Gathering everything into the master equation (11) and expanding the matrix convolution products within the equation, we have:

$$\begin{aligned}\frac{dp_i}{dt} &= \sum_{j=1}^{n}\sum_{k=1}^{n}\Big(\mathbf{T}_{ik}(t)\star\mathcal{L}^{-1}\{\widehat{\mathbf{D}}^{-1}(s)_{kk}\}-\delta(t)\delta_{ik}\Big)\star\mathbf{K}_{kj}(t)\star p_j(t)\\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\lambda_{ik}e^{-\Lambda_k t}\star\left(\delta(t)+\frac{\delta'(t)}{\Lambda_k}\right)-\delta(t)\delta_{ik}\right)\star\Lambda_j\delta(t)\delta_{kj}\star p_j(t)\\ &= \left[\sum_{j=1}^{n}\lambda_{ij}e^{-\Lambda_j t}\star\big(\delta(t)\Lambda_j+\delta'(t)\big)\star p_j(t)\right]-\Lambda_i p_i(t).\end{aligned}\tag{17}$$

Using integration by parts,

$$e^{-\Lambda_j t}\star\big(\Lambda_j\delta(t)+\delta'(t)\big)=\Lambda_j e^{-\Lambda_j t}+\int_0^t e^{-\Lambda_j(t-s)}\delta'(s)ds=\delta(t),$$

yielding the simplified master equation

$$\frac{dp_i}{dt}=\left[\sum_{j=1}^{n}\lambda_{ij}p_j(t)\right]-\Lambda_i p_i(t).\tag{18}$$

Making the qualitative assumption that $\lambda_{ij}=0$ when $i\nsim j$, the above equation becomes

$$\frac{dp_i}{dt}=\sum_{j\sim i}\lambda_{ij}\left(p_j(t)-p_i(t)\right).$$

Solving for the steady-state solution, one sets the L.H.S. equal to 0 identically; this becomes a Dirichlet problem for the graph Laplacian weighted by the intensities λ_{ij}. In the very interesting paper [7], Chung and Yau solve this problem (in a more general framework) using discrete Green's Functions.

As a final comment, note that setting $\lambda_{ij}=1$ when $i\sim j$ and $\lambda_{ij}=0$ otherwise, the problem here reduces to $-\Delta(\mathbf{p}(t))=0$, which we solved in the case of the asynchronous random walk on a static underlying graph. This is all to conclude than when one chooses the WTD to be uniformly Poisson on adjacent edges, the temporality of the edges becomes redundant (*think about it*).

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